Probability 2 - Notes 8

Transformations of variables

Let X and Y be jointly continuous random variables with joint p.d.f. $f_{X,Y}(x,y)$ which has support S contained in $\Re^2$. Consider random variables $U = g(X,Y)$ and $V = h(X,Y)$ which is a one to one mapping from S to D with inverses $X = a(U,V)$ and $Y = b(U,V)$ which have continuous partial derivatives. Then

$$f_{U,V}(u,v) = f_{X,Y}(a(u,v),b(u,v)) \times \left\| \begin{array}{cc} \frac{\partial a(u,v)}{\partial u} & \frac{\partial a(u,v)}{\partial v} \\ \frac{\partial b(u,v)}{\partial u} & \frac{\partial b(u,v)}{\partial v} \end{array} \right\|$$

for $(u,v) \in D$. You can find D by rewriting the constraints on the ranges of x and y in terms of u and v.

Example. $X \sim Exp(\theta)$ independent of $Y \sim Exp(\theta)$. Find the joint p.d.f. for $U = X+Y$ and $V = Y$.

The inverses are $X = U - V$ and $Y = V$. Also $f_{X,Y}(x,y) = \theta^2 e^{-\theta(x+y)}$ for $x > 0$ and $y > 0$. Hence using the result above:

$$f_{U,V}(u,v) = \theta^2 e^{-\theta u} \times \left\| \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right\| = \theta^2 e^{-\theta u}$$

The ranges $x > 0$ and $y > 0$ become $u - v > 0$ and $v > 0$, i.e. $0 < v < u < \infty$.

Student's t distribution Let $X \sim N(0,1)$ independent of $Y \sim \chi^2_n$. Find the joint p.d.f. for $T = \frac{X}{\sqrt{Y/n}}$ and $V = Y$ and hence find the marginal p.d.f. for T.

The inverses are $X = T\sqrt{V/n}$ and $Y = V$. Also

$$f_{X,Y}(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)\left(\frac{y^{(n/2)-1}e^{-y/2}}{2^{(n/2)}\Gamma(n/2)}\right)$$

for all x and for $y > 0$. Hence using the result above:

$$f_{T,V}(t,v) = \frac{v^{(n/2)-1}e^{-v(1+(t^2/n))/2)}}{\sqrt{\pi}2^{(n+1)/2}\Gamma(n/2)} \times \left\| \begin{array}{cc} \frac{\sqrt{v}}{\sqrt{n}} & \frac{t}{2\sqrt{nv}} \\ 0 & 1 \end{array} \right\| = \frac{v^{((n+1)/2)-1}e^{-v(1+(t^2/n))/2)}}{\sqrt{n}\sqrt{\pi}2^{(n+1)/2}\Gamma(n/2)}$$

The range $y > 0$ become $v > 0$, hence $-\infty < t < \infty$ and $0 < v < \infty$.

We find the p.d.f. for T by integrating out over V and using the result that a Gamma p.d.f. integrates to one.

$$
\begin{aligned}
f_T(t) &= \frac{\Gamma((n+1)/2)}{\sqrt{n}\sqrt{\pi}\Gamma(n/2)\left(1+(t^2/n)\right)^{(n+1)/2}} \int_0^\infty \frac{\left((1+(t^2/n))/2\right)^{(n+1)/2} v^{((n+1)/2)-1} e^{-v(1+(t^2/n))/2)}}{\Gamma((n+1)/2)} dv \\
&= \frac{\Gamma\left(\frac{(n+1)}{2}\right)}{\sqrt{n}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}
\end{aligned}
$$

for $-\infty < t < \infty$. Note that $\Gamma(1/2) = \sqrt{\pi}$ and the Beta function $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ so the p.d.f. of T is usually written as

$$
f_T(t) = \frac{1}{\sqrt{n} B\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}
$$

Fisher's F Distribution Let $X \sim \chi^2_n$ independent of $Y \sim \chi^2_m$. Find the joint p.d.f. for $U = \frac{X/n}{Y/m}$ and $V = Y$ and hence find the marginal p.d.f. for U.

The inverses are $X = (n/m)UV$ and $Y = V$. Also

$$
f_{X,Y}(x,y) = \frac{x^{(n/2)-1} y^{(m/2)-1} e^{-(x+y)/2}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)}
$$

for $x > 0$ and $y > 0$. Hence using the result above:

$$
\begin{aligned}
f_{U,V}(u,v) &= \frac{(n/m)^{(n/2)-1} u^{(n/2)-1} v^{((n+m)/2)-2} e^{-v(1+(nu/m))/2}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} \times \left\| \begin{array}{cc} (n/m)v & (n/m)u \\ 0 & 1 \end{array} \right\| \\
&= \frac{(n/m)^{n/2} u^{(n/2)-1} v^{((n+m)/2)-1} e^{-v(1+(nu/m))/2}}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)}
\end{aligned}
$$

The ranges $x > 0$ and $y > 0$ become $uv > 0$ and $v > 0$ and hence $u > 0$ and $v > 0$. Then

$$
\begin{aligned}
f_U(u) &= \frac{(n/m)^{n/2} u^{(n/2)-1}\Gamma((n+m)/2)}{\Gamma(n/2)\Gamma(m/2)\left(1+(nu/m)\right)^{(n+m)/2}} \int_0^\infty \frac{\left(1+(nu/m)\right)^{(n+m)/2} v^{((n+m)/2)-1} e^{-v(1+(nu/m))/2}}{2^{(n+m)/2}\Gamma((n+m)/2)} dv \\
&= \frac{(n/m)^{n/2} u^{(n/2)-1}}{B((n/2),(m/2))\left(1+(nu/m)\right)^{(n+m)/2}}
\end{aligned}
$$

Conditional Distributions. For each x for which $f_X(x) > 0$, we define the conditional p.d.f. for $Y|X = x$ by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

This is easily seen to be a p.d.f. for Y for each fixed value of x. Analogous results hold as for the discrete case:

1. $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx$

2. $E[g(Y)] = E[E[g(Y)|X]]$ and hence $E[Y] = E[E[Y|X]]$ and $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$.

3. $M_Y(t) = E[e^{tY}] = E[E[e^{tY}|X]]$.

4. For both the discrete and continuous case a similar result to (2) holds for the expectation of a function of both X and Y, namely $E[g(X,Y)] = E[E[g(X,Y)|X]]$.

There are existence requirements for the expectations, which we assume hold. A brief proof is given for (4) when X and Y are jointly continuous random variables.

Let A denote the set of real values x for which $f_X(x) > 0$. For $x \in A$,

$$E[g(X,Y)|X = x] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y) dy = \frac{1}{f_X(x)} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy$$

Then $E[g(X,Y)|X]$ is the function of X which takes value $E[g(X,Y)|X = x]$ when $X = x$ for all $x \in A$. Therefore

$$\begin{aligned}
E[E[g(X,Y)|X]] &= \int_{x \in A} E[g(X,Y)|X = x] f_X(x) dx \\
&= \int_{x \in A} f_X(x) \left(\frac{1}{f_X(x)} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy \right) dx \\
&= \int_{x \in A} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy dx \\
&= E[g(X,Y)]
\end{aligned}$$

Examples.

1. $f_{X,Y}(x,y) = 2$ for $x > 0$, $y > 0$ and $x + y < 1$. Then $f_X(x) = 2(1-x)$ for $0 < x < 1$ and hence $f_{Y|X}(y|x) = \frac{1}{1-x}$ for $0 < y < 1-x$. Hence $Y|X = x \sim U(0, 1-x)$.

2. $Y|X = x \sim N(a + bx, \sigma^2)$ and $X \sim N(\mu, \tau^2)$. We will first find $E[Y]$ and $Var(Y)$.

$E[Y|X] = a + bX$ and $Var(Y|X) = \sigma^2$. therefore $E[Y] = E[E[Y|X]] = E[a + bX] = a + b\mu$ and

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X]) = E[\sigma^2] + Var(a+bX) = \sigma^2 + b^2\tau^2$$

We will now find the m.g.f. for Y and hence obtain the distribution of Y. Now $E[e^{tY}|X]$ is just the m.g.f. of Y over the conditional distribution of $Y|X$, so $E[e^{tY}|X] = e^{(a+bX)t+\sigma^2(t^2/2)}$. Therefore

$$\begin{aligned} M_Y(t) &= E[E[e^{tY}|X]] = E\left[e^{(a+bX)t+\sigma^2(t^2/2)}\right] = e^{at+\sigma^2(t^2/2)} M_X(bt) \\ &= e^{at+\sigma^2(t^2/2)} e^{\mu(bt)+\tau^2((bt)^2/2)} = e^{(a+b\mu)t+(\sigma^2+b^2\tau^2)(t^2/2)} \end{aligned}$$

Hence $Y \sim N(a+b\mu, \sigma^2+b^2\tau^2)$.

Joint Moment Generating Functions.

$M_{X,Y}(s,t) = E[e^{sX+tY}]$. The properties are given below:

1. A uniqueness property holds as for the m.g.f. for a single random variable X. So if we recognise that the joint m.g.f. comes from a specific joint p.d.f., then X, Y have that joint p.d.f.

2. $M_X, Y(0,0) = 1$; $M_X(s) = M_{X,Y}(s,0)$, $M_Y(t) = M_{X,Y}(0,t)$. So you can then find the distribution, mean and variance for each of X and Y.

3. $\frac{\partial^2 M_{X,Y}(s,t)}{\partial s \partial t}$ evaluated at $s = t = 0$ gives $E[XY]$.

4. If X and Y are independent then $M_{X,Y}(s,t) = E[e^{sX}e^{tY}] = E[e^{sX}]E[e^{tY}] = M_X(s)M_Y(t)$.

5. If $M_{X,Y}(s,t) = g(s)h(t)$ then, from property 1, $M_X(s) = h(0)g(s)$, $M_Y(t) = g(0)h(t)$ and $1 = g(0)h(0)$. Hence $M_{X,Y}(s,t) = M_X(t)M_Y(t)$ and by result 4 and result 1, concerning the uniqueness of the joint m.g.f., X and Y are independent.