

## Probability 2 - Notes 6

**Jointly Continuous Random Variables**  $X$  and  $Y$  are jointly continuous random variables if there exists a function  $f_{X,Y}(x,y)$  (the joint p.d.f. ) which maps  $\mathcal{R}^2$  into  $[0,\infty)$  so that, for any measurable set  $A$ ,  $P((X,Y) \in A) = \int \int_{(x,y) \in A} f_{X,Y}(x,y) dx dy$ .

All sets of practical interest are measurable. The joint p.d.f. is non-negative for all entries. If we consider the 3-D plot of the joint p.d.f. as a function of the entries  $x, y$ , then  $P((X,Y) \in A)$  is just the volume below the p.d.f. with base the set  $A$ . Hence for any set  $A$  with area zero the corresponding probability is zero, e.g.  $P(X = x, Y = y) = 0$  and  $P(X + Y = c) = 0$ .

(i) Joint c.d.f.  $F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(s,t) dt ds$ . Note that the order of integration can be reversed.

(ii) From calculus (subject to differentiability constraints on the joint c.d.f.)  $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial y \partial x} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$ . For  $dx$  and  $dy$  small and positive,  $f_{X,Y}(x,y) dx dy \cong P(X \in (x - dx, x], Y \in (y - dy, y])$ .

(iii)  $1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(s,t) ds dt$ .

(iv)  $F_X(x) = F_{X,Y}(x, \infty)$ . Differentiating with respect to  $x$  gives the result that the marginal p.d.f.'s for  $X$  can be obtained by integrating the joint p.d.f. over the entry  $y$  for  $Y$ . i.e.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$$

An equivalent result holds for  $Y$ . Note that  $f_X(x)$  is just the cross-sectional area above the line  $X = x$ .

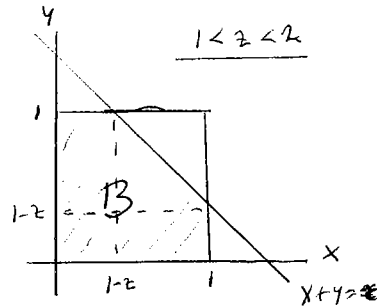
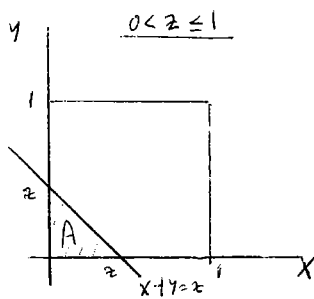
We find  $P((X,Y) \in A)$  by evaluating the double integral. You have done this in Calculus 2. Remember that when you specify  $f_{X,Y}(x,y)$  in the double integral it may be zero over part of the range for  $x$  and  $y$ , so make sure you put in the correct limits on the integrals corresponding to the only the values of  $(x,y)$  which are in the set  $A$  and are also in the support of the joint p.d.f.

**Special case with constant p.d.f.** Let  $f_{X,Y}(x,y) = C$  for  $(x,y) \in S$  and zero elsewhere. Then from (iii),  $1 = C \times (\text{area of } S)$ . Hence  $C = (\text{area } S)^{-1}$ . If  $A$  is a subset of  $S$ , then  $P((X,Y) \in A) = C(\text{area of } A) = \frac{\text{area } A}{\text{area } S}$ .

**Example.**  $f_{X,Y}(x,y) = C$  for  $0 < x < 1$  and  $0 < y < 1$ . The joint p.d.f. is zero elsewhere. Then  $C = 1$ . The cross-sectional area above the line  $X = x$  is just  $C \times 1$  if  $0 < x < 1$  and is zero elsewhere. Hence  $f_X(x) = 1$  for  $0 < x < 1$  and zero elsewhere.

Let  $Z = X + Y$ . We will find  $P(Z \leq z) = P(X + Y \leq z)$ . When  $z \leq 0$ ,  $P(Z \leq z) = 0$  and when  $z \geq 2$ ,  $P(Z \leq z) = 1$ . When  $0 < z \leq 1$  then  $P(Z \leq z)$  corresponds to  $P((X,Y) \in A)$  where  $A$  is just the interior of the triangle bounded by the  $X$  and  $Y$  axis and the line  $X + Y = z$ . The area of  $A$  is just  $z^2/2$ . Since  $C = 1$ ,  $P(Z \leq z) = z^2/2$  for  $0 < z \leq 1$ . Now consider  $1 < z < 2$ . Then  $P(Z \leq z)$  corresponds to  $P((X,Y) \in B)$  where  $B$  is the square forming the support of the joint p.d.f. but excluding the triangle formed by the lines  $X = 1$ ,  $Y = 1$  and  $X + Y = z$ . The area of this triangle is  $(1 - (1 - z))/2 = (2 - z)^2/2$ . Hence the area of  $B$  is  $(1 - (2 - z)^2/2)$  and so, since  $c = 1$ ,  $P(Z \leq z) = 1 - (2 - z)^2/2$  for  $1 < z < 2$ . Plots are on the next page.

Therefore we have found the c.d.f. for  $Z = X + Y$ . Differentiating with respect to  $z$  will give the p.d.f. So  $f_Z(z) = z$  for  $0 < z \leq 1$ ,  $f_Z(z) = (2 - z)$  for  $1 \leq z \leq 2$  and  $f_Z(z) = 0$  elsewhere.



**General case** You can find the probability of an event for  $X$  and  $Y$  by integrating in either order.

**Example.**  $f_{X,Y}(x,y) = Cxy$  for  $0 < x < y < 1$  and is zero elsewhere. We will find the marginal p.d.f.'s and obtain  $C$ . Note the dependent ranges. When we find  $f_X(x)$ ,  $x$  is held fixed and we integrate over the values of  $y$  for which the joint p.d.f. is positive, i.e. over  $x < y < 1$ .

$$f_X(x) = \int_x^1 Cxy dy = \left[ Cx \frac{y^2}{2} \right]_{y=x}^{y=1} = \frac{Cx(1-x^2)}{2}$$

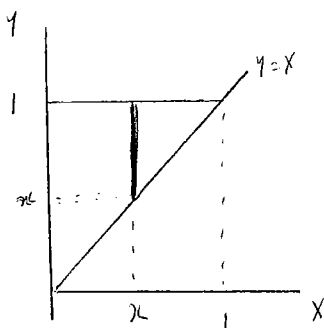
for  $0 < x < 1$  and  $f_X(x) = 0$  elsewhere. Integrating the p.d.f. for  $X$  identifies  $C$ .

$$1 = \int_0^1 \frac{C}{2}(x-x^3) dx = \frac{C}{2} \left( \frac{1}{2} - \frac{1}{4} \right) = C/8$$

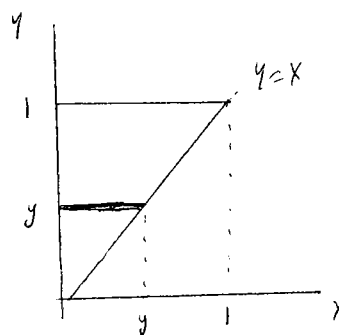
Therefore  $C = 8$ . When we find  $f_Y(y)$ ,  $y$  is held fixed and we integrate over the values of  $x$  for which the joint p.d.f. is positive, i.e. over  $0 < x < y$ .

$$f_Y(y) = \int_0^y 8xy dx = \left[ 8y \frac{x^2}{2} \right]_{x=0}^{x=y} = 4y^3$$

for  $0 < y < 1$  and  $f_Y(y) = 0$  elsewhere.



Finding  $f_X(z)$



Finding  $f_Y(y)$