Simon and the Monster

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Simon P. Norton (1952 – 2019)

Simon in 2003
(Image courtesy of Alexander Ivanov)

Much of Simon's mathematical research concerned the Monster group, a fascinating and important mathematical object. But what is the Monster?

What is a group?

For our purposes, a **group** G is a set of **operations**, satisfying the following properties:

- if X and Y are operations in G, then so is the operation of doing X followed by Y (which is denoted by XY)
- if X, Y, Z are operations in G, then $(XY)Z = X(YZ)$
- the operation of "doing nothing", the **identity operation** denoted I, is in G, satisfying $IX = XI = X$ for all X in G
- if X is an operation in G then there is an operation X^{-1} in G such that $XX^{-1} = X^{-1}X = I$.

The set of symmetry operations of an object forms a group, so groups play important roles in geometry, networks, physics, chemistry, biology, and indeed anywhere symmetry occurs.

A group of symmetries of the projective plane of order 3

Consider the finite geometrical object $\mathbb{P}_3$, the **projective plane of order 3**, having 13 **points**:

$$0, 1, \ldots, 12$$

and 13 **lines**:

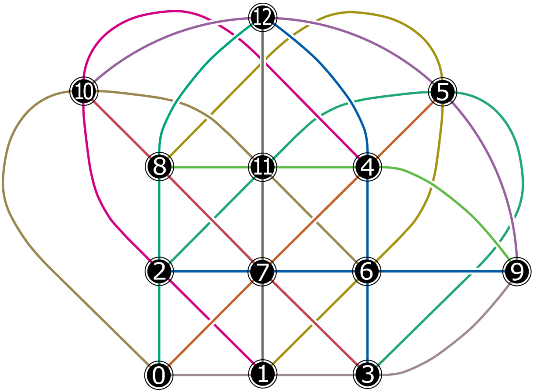
$\{0,1,3,9\}$, $\{1,2,4,10\}$, $\{2,3,5,11\}$, $\{3,4,6,12\}$, $\{0,4,5,7\}$, $\{1,5,6,8\}$, $\{2,6,7,9\}$, $\{3,7,8,10\}$, $\{4,8,9,11\}$, $\{5,9,10,12\}$, $\{0,6,10,11\}$, $\{1,7,11,12\}$, $\{0,2,8,12\}$.

Now note that applying the operation A which replaces 0 by 1, 1 by 2, 2 by 3, ..., 12 by 0, leaves the sets of points and lines of $\mathbb{P}_3$ unchanged, that is, A is a symmetry of $\mathbb{P}_3$.

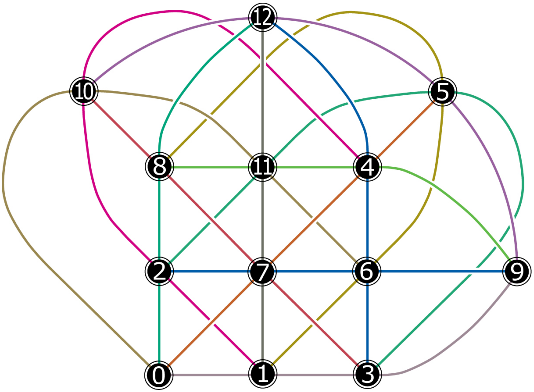
Thus, also $A^2 = AA$, $A^3 = AAA$, ..., $A^{13} = I$ are symmetries of $\mathbb{P}_3$, and it follows that $\{I, A, A^2, \ldots, A^{12}\}$ is a group of symmetries of $\mathbb{P}_3$ (for example, $A^{-1} = A^{12}$).

A drawing of $\mathbb{P}_3$

Here is the projective plane $\mathbb{P}_3$ of order 3 drawn by G. Eric Moorhouse. I am very grateful to him for providing this. For clarity, each line is given a distinct colour.



Another symmetry of $\mathbb{P}_3$



You can check that the operation B which interchanges the points 0 and 1, 4 and 8, 6 and 7, and 10 and 12, and fixes each of the remaining points, is also a symmetry of $\mathbb{P}_3$. Note that $B^2 = I$.

The group L of all symmetries of $\mathbb{P}_3$

Now **all** expressions in A and B, such as A^3BA^2BA, BA^4BA^{-1}, etc. are symmetries of $\mathbb{P}_3$. The set L of all these expressions forms the **group generated** by A and B, and this is denoted by $L = \langle A, B \rangle$.

It turns out that L has exactly 5,616 distinct operations (we say that L has **order** 5,616), and that L is the group of **all** the symmetries of $\mathbb{P}_3$.

The group $\langle A \rangle = \{I, A, A^2, \ldots, A^{12}\}$ we have seen before has order 13, and is a **subgroup** of L (since it is a subset of L forming a group).

What is a simple group?

Suppose G is a group with more than one operation and H is a subgroup of G. We say that H is a **normal** subgroup of G if $g^{-1}hg$ is an operation in H for every h in H and every g in G.

For example, the **trivial subgroup** $\{I\}$ of G, consisting only of the identity operation, and the whole group G itself are both normal subgroups of G. If these are the only normal subgroups of G, we say that G is a **simple** group.

The finite simple groups are the building blocks of **all** finite groups, in a somewhat analogous way that the prime numbers are the building blocks of all positive whole numbers.

Many problems about finite groups reduce to problems about finite simple groups, so we would like to know as much as possible about these.

The classification of finite simple groups

One of the great achievements of 20th century mathematics was the classification of finite simple groups, combining the efforts of about 100 mathematicians, with a proof taking over 10,000 pages! This proof is presently being revised and simplified, but will still take more than 5,000 pages.

It turns out that the finite simple groups consist of

- the groups of prime order (these are precisely the simple groups with the property that $XY = YX$ for all operations X and Y in the group, such as the group $\langle A \rangle$ of order 13 we have seen)
- seventeen further well-defined infinite families (one of which includes the group L, better known as $L_3(3)$ or $\mathrm{PSL}(3,3)$, of symmetries of the projective plane of order 3)
- 26 **sporadic** simple groups which fall into no infinite family and form no family of their own.

The Monster

The largest, and most important sporadic simple group is the Fischer-Griess Monster, or simply the **Monster**, denoted $\mathbb{M}$, which has order

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71 \approx 8 \times 10^{53}$$

and which involves 20 of the 26 sporadic simple groups (including itself), as "building blocks" of subgroups.

More on the Monster

Strong evidence for the existence of the Monster was discovered independently in 1973 by Bernd Fischer in Germany and Robert Griess in America.

Much work on properties of the (hypothetical) Monster was then undertaken by Fischer, Griess, John Thompson, John Conway, Simon Norton, Koichiro Harada, Donald Livingstone, Michael Thorne, and others. In particular, it was shown that if the Monster group existed, it would have to contain two further previously unknown sporadic simple groups as subgroups.

Finally, in 1982, in a tour-de-force, Griess published an 102-page paper constructing the Monster by hand as a group of symmetries of a certain 196,884-dimensional algebra over the rational numbers. In this paper, Griess credits work of Simon Norton as the inspiration for the construction.

Simon's PhD Thesis

Simon completed his PhD thesis "*F* and Other Simple Groups" in 1975, in which he investigated and constructed certain groups which would be subgroups of the (then hypothetical) Monster.

Around the same time the putative Monster group was being investigated, Harada was investigating a possible new finite simple group F with an "involution centralizer" of shape $(2{\cdot}HS)$:2, where HS is the Higman-Sims sporadic simple group.

This group F looked to be the same group as one of the two new sporadic simple groups which would have to be subgroups of the Monster (assuming its existence).

The Harada-Norton group

Simon studied the putative group F, and constructed (with computational assistance from Peter Smith) a group with its properties as a linear group permuting a certain set of 1,140,000 vectors in a 133-dimensional vector space over the real numbers.

He proved that the group so constructed is a simple group of order $2^{14}.3^6.5^6.7.11.19$, and that it is the unique simple group having the properties of F determined by Harada.

The sporadic simple group F is now called the Harada-Norton group, and is denoted by HN.

Monstralizers

After his PhD, Simon continued to work on the structure of the Monster.

He coined the term **monstralizer** of a subgroup H of $\mathbb{M}$ to mean the **centralizer** in $\mathbb{M}$ of H, which is

$$C_{\mathbb{M}}(H) = \{g \text{ in } \mathbb{M} \mid gh = hg \text{ for every } h \text{ in } H\}.$$

For example, the monstralizer of the Harada-Norton group HN is a "dihedral group" D_{10} of order 10.

According to Simon, "large subgroups of the Monster are characterized by their monstralizers, and any containment between subgroups is likely to be 'explained' by a containment (the other way around) between their monstralizers". For example, the monstralizer of HN shows the containment of HN in the monstralizer of a certain subgroup of order 2 of $\mathbb{M}$, and hence in the so-called "Baby Monster" group.

The Anatomy of the Monster

Simon classified and studied certain ordered pairs (G_1, G_2) of subgroups of $\mathbb{M}$, such that G_1 is the monstralizer of G_2, and G_2 is the monstralizer of G_1.

He published a summary of this work together with further results on the structure of $\mathbb{M}$ as "The Anatomy of the Monster: I" in 1998.

In 2002, Simon and Robert Wilson published "The Anatomy of the Monster: II", giving the current state of the classification of the maximal subgroups of the Monster. (A **maximal** subgroup of a group G is a subgroup properly contained in G, but not properly contained in any other subgroup.)

Some recent developments

Martin Seysen has developed a computer package, officially released in 2024, providing amazingly rapid multiplication of operations in the Monster, and extremely compact storage of such operations.

This method makes use of the action of $\mathbb{M}$ on Conway's modified version of the Griess algebra, which Simon had also worked on.

This new, very highly efficient method of computing in $\mathbb{M}$ was used by Heiko Dietrich, Melissa Lee, Anthony Pisani, and Tomasz Popiel to build on the large body of work by Simon, Robert Wilson, Petra Holmes, and others, to complete the classification of the maximal subgroups of the Monster and to construct them explicitly.

I think that Simon, who was very interested in explicit computation in the Monster, would be impressed and pleased!

The ATLAS project

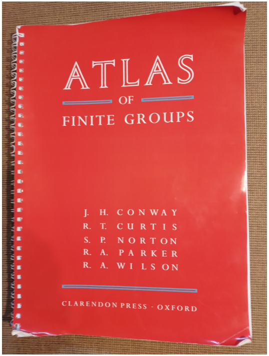
The ATLAS project was started in the early 1970s by John Conway, who obtained a grant and hired his former PhD student, Robert Curtis, to create a handbook listing "all interesting properties of all interesting groups".

In due course, Simon became a key member of the ATLAS team, followed by Richard Parker and Robert Wilson.

I was very lucky to be able to make use of the valuable information in prepublication versions of the ATLAS, together with abundant help from ATLAS authors, during my PhD from 1981 to 1985, and to be able to provide many of the group presentations by generators and relations from my PhD thesis for inclusion in the published ATLAS.

The ATLAS of Finite Groups

The ATLAS of Finite Groups, or simply the ATLAS, was published in 1985, and contains character tables, constructions, and many complete lists of maximal subgroups for a total of 93 finite simple groups (including all 26 sporadic simple groups), and closely related groups, with a full 15 pages devoted to the Monster.



This year we celebrate the 40th anniversary of the publication of the ATLAS.

Moonshine

One aspect of the Monster which deserves a Simon Norton Lecture of its own is **Monstrous Moonshine**, the amazing and mysterious connection between the Monster group and certain classical functions of complex analysis, important in many areas, such as number theory and mathematical physics.

Evidence for Moonshine was discovered by John McKay, and precise conjectures were developed and published by John Conway and Simon Norton in 1979.

The proof of these conjectures, using ideas from string theory, helped to earn Richard Borcherds a Fields Medal (the highest honour for a mathematician) in 1998.

The Bimonster group $\mathbb{B}$

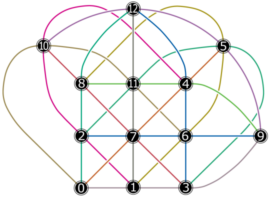
The **Bimonster** group $\mathbb{B}$ is generated by two copies of the Monster commuting with each other, together with an operation of order 2 interchanging the copies:

$$\mathbb{B} = (\mathbb{M} \times \mathbb{M}){:}2.$$

The Bimonster appears to be a natural object, and has arisen in certain areas of mathematics, but we don't really know why.

The projective plane generators and relations

Using his knowledge of the structure of the Monster, Simon found a special set S of 26 operations $s_0, s_1, \ldots, s_{12}, s_{\{0,1,3,9\}}, s_{\{1,2,4,10\}}, \ldots, s_{\{0,2,8,12\}}$ in the Bimonster, corresponding to the 13 points and 13 lines of the projective plane of order 3,



, generating the Bimonster and satisfying the following **projective plane relations**:

- $s^2 = I$, for each s in S
- $st = ts$ if s, t in S both correspond to points or both correspond to lines, or if one of s and t corresponds to a line and the other to a point **not on** the line
- $sts = tst$ if one of s, t in S corresponds to a line and the other to a point **on** the line.

The group Y^*

In the mid-1980s, after my PhD, Conway (in England and America), Simon (in England), and I (in Canada) took a closer look at the projective plane generators and relations in the Bimonster.

We started with a small list of axioms (later simplified by Conway and Arvan Pritchard), satisfied by the Bimonster, and proved that any group G satisfying these axioms would be a quotient of a specific group Y^*, which was generated by 26 operations satisfying the projective plane relations above, together with certain further defining relations.

We also showed that the group $L_3(3)$:2 of all symmetries of the projective plane relations could be realized by symmetries of Y^*.

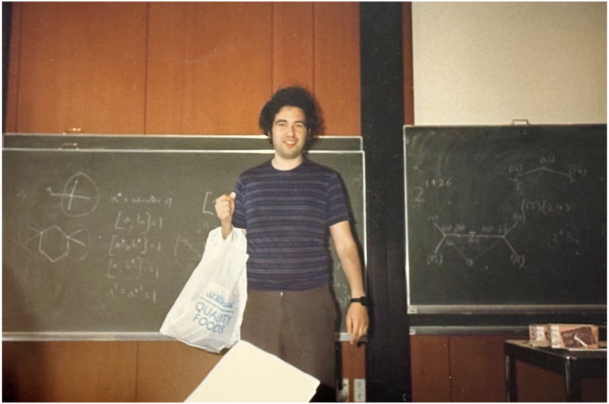
We then classified the subgroups of Y^* generated by the "connected" subsets of the 26 projective plane generators of Y^*, illustrating many subgroups of the Bimonster and the Monster, and relationships between these subgroups.

$Y^* = \mathbb{B}$

In due course, in the later 1980s, Simon answered a question of mine by finding the subgroup of shape $2^{1+24}.Co_1$ of the Monster in the group Y^*.

This then allowed Simon, building on his previous work and that of Conway, Steve Linton, myself, and others, to apply a characterization of the Monster due to Alexander Ivanov, to prove in 1990 that Y^* must be the Bimonster itself.

Here is Simon around 1987 at Oberwolfach in Germany. It looks like he is lecturing about finding $2^{1+26}.(2^{24}{:}Co_1)$ in terms of the projective plane generators for Y^*.



I am very grateful to Alexander Ivanov for this image.

A presentation for the Bimonster

In 1988, I proved that the group Y^* (later shown to be the Bimonster) is the group defined by generators

$$s_0, s_1, \ldots, s_{12}, \quad s_{\{0,1,3,9\}}, s_{\{1,2,4,10\}}, \ldots, s_{\{0,2,8,12\}}$$

corresponding to the points and lines of the projective plane of order 3, subject (only) to the projective plane relations together with the single additional relation

$$(s_0 s_{\{1,2,4,10\}} s_1 s_{\{0,2,8,12\}} s_2 s_{\{0,1,3,9\}})^4 = I.$$

Simon in 2013

Simon continued to be interested in the Monster, even as his focus shifted to public transport campaigning.

Here is Simon pictured front and centre with fellow mathematicians at a workshop at Imperial College, London in 2013.

The final word

The final word goes to Simon via the title of his 2017 paper from a conference in Princeton in 2015 celebrating John Conway, the 30th anniversary of the publication of the ATLAS of Finite Groups, and the 20th anniversary of the publication of the ATLAS of Brauer Characters.

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The Monster is fabulous

Simon P. Norton

ABSTRACT. We summarize some work, nostly done around the time the ATLAS was published, that we hope will form part of a better understanding of why the Monster exists.

1. Introduction

The conference in whose proceedings this paper appears was organised to celebrate 30 years of the ATLAS of Finite Groups [**1**], and as a launchpad for Siobhan Roberts's biography of the lead author, John Conway [**8**]. It is for both these reasons that we feel that it is appropriate to bring together in one place a summary of work related to the discovery of a relation between the Fischer-Griess Monster $\mathbb{M}$ and the projective plane of order 3.