Strongly regular graphs

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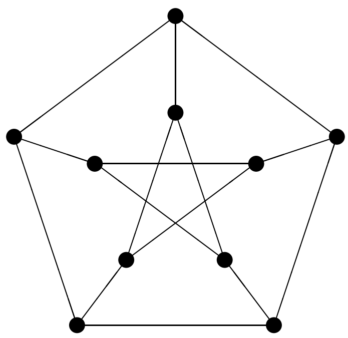
Abstract

Strongly regular graphs form an important class of graphs which lie somewhere between the highly structured and the apparently random. This chapter gives an introduction to these graphs with pointers to more detailed surveys of particular topics.

1 An example

Consider the Petersen graph:



Of course, this graph has far too many remarkable properties for even a brief survey here. (It is the subject of a book [17].) We focus on a few of its properties: it has ten vertices, valency 3, diameter 2, and girth 5. Of course these properties are not all independent. Simple counting arguments show that a trivalent graph with diameter 2 has at most ten vertices, with equality if and only it has girth 5; and, dually, a trivalent graph with girth 5 has at least ten vertices, with equality if and only it has diameter 2.

The conditions "diameter 2 and girth 5" can be rewritten thus: two adjacent vertices have no common neighbours; two non-adjacent vertices have exactly one common neighbour. Replacing the particular numbers 10, 3, 0, 1 here by general parameters, we come to the definition of a strongly regular graph:

Definition A *strongly regular graph* with parameters (n, k, λ, μ) (for short, a $\mathrm{srg}(n, k, \lambda, \mu)$) is a graph on n vertices which is regular with valency k and has the following properties:

- any two adjacent vertices have exactly λ common neighbours;
- any two nonadjacent vertices have exactly μ common neighbours.

So the Petersen graph is a $\mathrm{srg}(10, 3, 0, 1)$.

The complete and null graphs are vacuously strongly regular, the parameters μ and λ respectively being undefined for them. Often these trivial cases are excluded.

The four parameters are not independent. Choose a vertex v; counting in two ways the ordered pairs (x, y) of adjacent vertices such that x is adjacent to v but y is not, we obtain the following result.

Proposition 1.1 *The parameters* (n, k, λ, μ) *of a strongly regular graph satisfy the equation*

$$k(k - \lambda - 1) = (n - k - 1)\mu.$$

Later in this chapter we will see that the parameters also satisfy various algebraic conditions and inequalities. A complete characterisation of the parameter sets of strongly regular graphs is not known.

This section concludes with two basic facts about strongly regular graphs.

Proposition 1.2 *(a) The complement of a strongly regular graph is also strongly regular.*

(b) *A strongly regular graph is disconnected if and only if it is isomorphic to* mK_r *(the disjoint union of* m *copies of* K_r*) for some positive integers* m *and* r*; this occurs if and only if* $\mu = 0$.

(c) *A connected strongly regular graph has diameter* 2.

2 A hierarchy of regularity conditions

We can put the definition of a strongly regular graph into a more general context as follows. For a non-negative integer t, let $C(t)$ be the following graph property:

> Let S_1 and S_2 be sets of at most t vertices. If the induced subgraphs on S_1 and S_2 are isomorphic, then the number of vertices joined to every vertex in S_1 is equal to the number joined to every vertex in S_2.

A graph satisfying $C(t)$ is sometimes called *t-tuple regular*. If G is such a graph, its *parameters* are the numbers $\lambda(S)$, where $\lambda(S)$ denotes the number of common neighbours of a set of vertices inducing a subgraph isomorphic to S, while S runs over all isomorphism types of graphs on at most t vertices.

The conditions $C(t)$ obviously become stronger as t increases. $C(0)$ is vacuous, and $\lambda(\emptyset)$ is just the number of vertices of the graph G. A graph satisfies $C(1)$ if and only if it is regular; $\lambda(\text{vertex})$ is the valency. A graph satisfies $C(2)$ if and only if it is strongly regular; $\lambda(\text{edge})$ and $\lambda(\text{nonedge})$ are the parameters called λ and μ in the last section.

In fact the hierarchy is finite [7]:

Theorem 2.1 *A graph which satisfies* $C(5)$ *also satisfies* $C(t)$ *for all non-negative integers* t*. The only such graphs are* nK_r *and its complement for all* $n, r \geq 1$*, the* 5*-cycle* C_5*, and the* 3×3 *square lattice* $L(K_{3,3})$*.*

There are only two known examples (up to complementation) of graphs satisfying $C(4)$ but not $C(5)$, the *Schläfli graph* on 27 vertices and the *McLaughlin graph* on 275 vertices. Infinitely many additional graphs satisfying $C(3)$ are known; all of them except for $L(K_{n,n})$ for $n \geq 4$ and finitely many others are associated with geometric objects such as quadrics in projective spaces and extremal generalised quadrangles.

On the other hand, there is no shortage of graphs satisfying $C(0)$ or $C(1)$. The number of graphs on n vertices is asymptotic to $2^{n(n-1)/2}/n!$, while the number of graphs of valency k is asymptotically $c_k n^{n(k-2)/2}/n!$ for $2 < k = o(\sqrt{n})$. (Estimates exist also for $k \sim cn$. See Wormald [32].) For both graphs and regular graphs, there are well-developed theories of random objects, including the assertion that almost all have no non-trivial automorphisms (explaining the $n!$ in the denominators of the asymptotic formulae).

Strongly regular graphs stand on the cusp between the random and the highly structured. For example, there is (up to isomorphism) a unique $\mathrm{srg}(36, 10, 4, 2)$; but a computation by McKay and Spence [20] showed that the number of $\mathrm{srg}(36, 15, 6, 6)$s is 32548. The pattern continues: there is a unique $\mathrm{srg}(m^2, 2(m-1), m-2, 2)$, but more than exponentially many $\mathrm{srg}(m^2, 3(m-1), m, 6)$s, as we will see.

This suggests that no general asymptotic results are possible, and that, depending on the parameters, strongly regular graphs can behave in either a highly structured or an apparently random manner.

Another role of strongly regular graphs is as test cases for graph isomorphism testing algorithms. The global uniformity ensured by the definition makes it harder to find a canonical labelling, while the superexponential number of graphs means that they cannot be processed as exceptions.

The Paley graphs and other strongly regular (and similar) graphs have been used as models of "pseudo-random graphs" (see Thomason [28]).

Recently, Fon-Der-Flaass [14] has observed that an old construction of Wallis [29] gives rise to more than exponentially many strongly regular graphs with various parameter sets to be discussed below. He also used these graphs to establish the following result about universality of strongly regular graphs:

Theorem 2.2 *Any graph on n vertices is an induced subgraph of a strongly regular graph on at most $4n^2$ vertices. This is within a constant factor of best possible.*

It is not known whether such a universality result holds for graphs satisfying $C(3)$.

3 Parameter conditions

The parameters of strongly regular graphs satisfy a number of restrictions. Some of the more important are described here.

Theorem 3.1 *Suppose that G is a strongly regular graph with parameters v, k, λ, μ. Then the numbers*

$$f, g = \tfrac{1}{2}\left(v - 1 \pm \frac{(v-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right)$$

are non-negative integers.

Proof Let A be the adjacency matrix of G. The fact that G is strongly regular shows that

$$A^2 = kI + \lambda A + \mu(J - I - A),$$

where J is the all-1 matrix. The all-1 vector j is an eigenvector of A with eigenvalue 1. Any other eigenvector of A is orthogonal to j, so the corresponding eigenvalue satifies the quadratic equation

$$x^2 = k + \lambda x + \mu(-1 - x).$$

From this we can calculate the two eigenvalues r, s and (using the fact that the trace of A is zero) their multiplicities f, g, finding the given expressions.

On the basis of this theorem, we can classify strongly regular graphs into two types:

- *Type I* or *conference graphs* have $(v-1)(\mu-\lambda) - 2k = 0$. This implies that $\lambda = \mu - 1$, $k = 2\mu$, and $n = 4\mu + 1$. (These are precisely the strongly regular graphs which have the same parameters as their complements.) It is known that they exist only if v is the sum of two squares.
- *Type II graphs*: for these graphs, $(\mu-\lambda)^2 + 4(k-\mu)$ is a perfect square, say d^2, where d divides $(v-1)(\mu-\lambda) - 2k$ and the quotient is congruent to $v - 1$ mod 2.

Examples of conference graphs include the *Paley graphs* $P(q)$: the vertex set of $P(q)$ is the finite field $\mathrm{GF}(q)$, where q is a prime power congruent to 1 mod 4, and u and v are adjacent if and only if $u-v$ is a non-zero square in $\mathrm{GF}(q)$ (Paley [23]).

The "non-principal" eigenvalues r and s of a Type II strongly regular graph are integers with opposite signs. The parameters may be conveniently expressed in terms of the eigenvalues as follows:

$$\lambda = k + r + s + rs, \qquad \mu = k + rs.$$

Of the other conditions satisfied by the parameters of a strongly regular graph, the most important is the *Kreĭn condition*, first proved by Scott [24] using a result of Kreĭn [18] from harmonic analysis. It states that

$$(r+1)(k+r+2rs) \le (k+r)(s+1)^2,$$

and a similar inequality with r and s reversed. The first bound is attained by a graph if and only if the second is attained by its complement. As we will see, the two inequalities are associated with the geometry of the two non-trivial eigenspaces of the adjacency matrix.

Some parameter sets satisfy all known necessary conditions. We mention a few of these here.

Pseudo-Latin square $PL_r(n)$, with $1 \le r \le n$: these have $v = n^2$, $k = r(n-1)$, $\lambda = r^2 - 3r + m$, $\mu = r(r-1)$. The significance of the name will appear in the next section.

Negative Latin square $NL_r(n)$, obtained by replacing r and n by their negatives in the formulae just given. (Since this gives $\lambda = r^2 + 3r - n$, we must have $n \ge r(r+3)$; equality holds if and only if the Kreĭn bound is met.)

Smith graphs whose somewhat involved parameters will not be given here (see [9], p. 111). These parameters always attain the Kreĭn bound.

These parameters arise in the theorem of Cameron *et al.* [8]:

Theorem 3.2 *Let G be a graph satisfying $C(3)$. Then either G is the pentagon, or its parameters are of pseudo-Latin square, negative Latin square, or Smith type.*

4 Geometric graphs

The notion of a partial geometry was introduced by Bose [3] as a tool for studying strongly regular graphs. Subsequently, partial geometries have been studied in their own right, and the concept has been extended in various ways, not all related to strongly regular graphs. This section focuses only on the connections.

A *partial geometry* with parameters (s,t,α), or $\mathrm{pg}(s,t,\alpha)$, is an incidence structure of points and lines satisfying the following axioms:

- any line contains $s+1$ points, and any point lies on $t+1$ lines;
- two lines meet in at most one point (or equivalently, two points lie on at most one line);
- if the point p is not on the line L, there are precisely α incident pairs (q,M), where q is a point of L and M a line through p.

Note that Bose used slightly different parameters: he put R, K, T for what are here called $s+1$, $t+1$, α.

The *dual* of a $\mathrm{pg}(s,t,\alpha)$ is obtained by interchanging the names "point" and "line" for the two types of object and dualising the incidence relation. It is also a partial geometry with parameters (t,s,α).

The *point graph* of a partial geometry is the graph whose vertices are the points of the geometry, adjacency being defined by collinearity. The *line graph* is the point graph of the dual geometry; that is, its vertices are the lines, and adjacency is given by concurrence.

Proposition 4.1 *The point graph of a* $\mathrm{pg}(s,t,\alpha)$ *is an* $\mathrm{srg}(n, s(t+1), s-1+t(\alpha-1), (t+1)\alpha)$, *where* $n=(s+1)(st+\alpha)/\alpha$.

The proof of this result is straightforward. Motivated by this, we say that a strongly regular graph G is *geometric* if it is the point graph of a partial geometry, and that G is *pseudo-geometric* if its parameters have the form given in Proposition 4.1 for some positive integers s,t,α. Sometimes we append the triple (s,t,α) to the term "geometric" or "pseudo-geometric".

Not every pseudo-geometric graph is geometric. Indeed, a pseudo-geometric (s,t,α) graph is geometric if and only if there is a collection L of $(s+1)$-cliques with the property that every edge lies in just one clique of L. If some edge lies in no $(s+1)$-clique, the graph is clearly not geometric;

but if there are "too many" cliques, it is often not clear whether a suitable collection can be selected.

The major result of Bose can now be stated:

Theorem 4.2 *Suppose that s, t, α are positive integers satisfying*

$$s > \tfrac{1}{2}(t+2)(t-1+\alpha(t^2+1)).$$

Then any pseudo-geometric (s, t, α) graph is geometric. Indeed, every edge of such a graph lies in a unique $(s+1)$-clique.

In order to see the power of this theorem, we look at partial geometries a little more closely. We divide them into six types.

Linear spaces, the case $\alpha = s+1$. A point not on a line L is collinear with every point of L. It follows that any two points lie on a (necessarily unique) line. Such structures are also known as 2-designs, pairwise balanced designs, or Steiner systems. The point graph of a linear space is just a complete graph and so of no interest.

We note in passing the asymptotic existence theorem of Richard Wilson [30]. A necessary condition for a linear space with n points having $s+1$ points on each line is that s divides $n-1$ and $s+1$ divides $n(n-1)$. In terms of s and t, this is the single condition that $s+1$ divides $t(t+1)$. Wilson showed the existence of a function $f(s)$ such that, if $t > f(s)$ and the necessary condition is satisfied, then a linear space exists.

Dual linear spaces, the case $\alpha = t+1$. The geometries are the duals of those in the preceding case, but the graphs (the line graphs of linear spaces) are much more interesting.

We examine two special cases. A linear space with two points on each line is just a complete graph K_n, and its line graph (the point graph of the dual) is just the usual line graph $L(K_n)$. We have $s = 1$, $\alpha = 2$, and $t = n-2$; the inequality in Bose's Theorem reduces to $n > 8$. Hence:

Corollary 4.3 *The graph $L(K_n)$ is strongly regular, and for $n > 8$ it is the unique strongly regular graph with its parameters.*

The conclusion actually holds for all $n \neq 8$. For $n = 8$, Chang [10] showed that there are exactly four strongly regular graphs with parameters

$(28, 12, 6, 4)$. Note that $L(K_n)$ is called the *triangular graph* $T(n)$ in the literature on strongly regular graphs.

A linear space with three points on each line is a *Steiner triple system*, or STS. The fact that an STS with n points exists for all n congruent to 1 or 3 mod 6 goes back to Kirkman in 1847. More recently, Wilson [30] showed that the number $v(n)$ of Steiner triple systems on an admissible number n of points satisfies

$$v(n) \geq \exp(n^2 \log n/6 - cn^2).$$

Moreover, a STS of order $n > 15$ can be recovered uniquely from its line graph. Hence there are superexponentally many $\mathrm{srg}(n, 3s, s+3, 9)$, for $n = (s+1)(2s+3)/3$ and s congruent to 0 or 2 mod 3.

Transversal designs, the case $s = \alpha$. In this case, it can be shown that there is a partition of the points into subsets traditionally called "groups" (though there is no connection with the algebraic notion): each line is a transversal for the family of groups, and any two points in different groups lie on a line. Thus, the point graph is complete multipartite, the partite sets being the groups.

Dual transversal designs, the case $t = \alpha$.

For $t = 2$, the graph is the line graph of the complete bipartite graph $K_{n,n}$. Bose's theorem gives us the following result:

Corollary 4.4 *The graph $L(K_{n,n})$ is strongly regular, and for $n > 4$ it is the unique strongly regular graph with its parameters.*

Again this holds for all $n \neq 4$. Shrikhande showed that there are just two strongly regular graphs with parameters $(16, 6, 2, 2)$, namely $L_{4,4}$ and one other now called the *Shrikhande graph* (defined below). In the literature on strongly regular graphs, $L(K_{n,n})$ is referred to as the *square lattice graph* $L_2(n)$.

If $t > 2$, for reasons which will become clear, we use new parameters n and r, where $n = s + 1$ and $r = t + 1$. The "groups" dualise to become a partition of the lines into "parallel classes", each parallel class forming a partition of the points. There are k lines in each parallel class, with n points on each, so the number of points is n^2. The geometry is called a *net* of order n and degree r.

Select two parallel classes $\{V_1, \ldots, V_n\}$ and $\{H_1, \ldots, H_n\}$. Now the points can be represented as a $n \times n$ grid, where the lines V_i run vertically and the H_j horizontally, and the unique point on V_i and H_j can be labelled p_{ij}.

Now let $\{L_1, \ldots, L_n\}$ be another parallel class of lines. Construct a $n \times n$ array Λ, with (i, j) entry l if $p_{ij} \in L_l$. It is clear that Λ is a Latin square of order n. By reversing the construction, any Latin square of order n gives rise to a net of order n and degree 3. Since the number of Latin squares of order k is asymptotic to $\exp(n^2 \log n - 2n^2)$, we obtain superexponentially many strongly regular graphs with these parameters.

There are just two non-isomorphic Latin squares of order 4, namely the Cayley tables of the Klein group and the cyclic group of order 4. They give rise to two non-isomorphic strongly regular graphs with parameters $(16, 9, 4, 6)$, whose complements are $L_2(4)$ and the Shrikhande graph.

If $r > 3$, we have $r - 2$ additional parallel classes, giving rise to $r - 2$ Latin squares. It is also easily checked that these Latin squares are *mutually orthogonal*, in the sense that given any two squares Λ and Λ', then for any given entries l, l', there is a unique cell (i, j) in which Λ and Λ' have entries l and l' respectively. Conversely, a set of $r - 2$ mutually orthogonal Latin squares (or *MOLS*, as they are called) of order n gives a net of order n and degree r, and hence a strongly regular graph.

For this reason, the point graph of a net of order n and degree r is called a *Latin square graph*, denoted $L_r(n)$. A pseudo-geometric graph with the parameters of $L_r(n)$ is the same as what was defined as a pseudo-Latin square graph $PL_r(n)$ earlier – hence the name.

If n is a prime power, then there exists a set of MOLS of order n of the maximum possible size, namely $n - 1$. (The corresponding net is an *affine plane* of order n. Choosing all subsets of $r-2$ of these squares, where $r \sim cn$ for $0 < c < 1$, we obtain again many strongly regular graphs with the same parameters (but only a fractional exponential number, in this case).

Generalised quadrangles, the case $\alpha = 1$. In this case the geometry is trivially recoverable from its point graph, since an edge lies in a unique maximal clique. There are "classical" GQs, related to classical groups (symplectic, unitary and orthogonal groups) much as projective planes are related to the projective groups $\mathrm{PGL}(3, q)$, and some non-classical examples, including some with non-classical parameters. Van Maldegham [21] surveys these geometries.

Fon-Der-Flaass [14] pointed out that some of Wallis' graphs [29], and some variants of them, have pseudo-geometric parameters corresponding to GQs with $s = q+1$, $t = q-1$, or with $s = t = q$, or with $s = q-1$, $t = q+1$, for prime powers q. So there are superexponentially many graphs for these parameter sets.

The rest, with $1 < \alpha < \min\{s,t\}$. Some examples are known but there is much less theory.

We conclude this section with a reference to the work of Neumaier [22], which improves Bose's classical results. From Neumaier's work, we quote two of his most notable results:

Theorem 4.5 *A strongly regular graph having parameters (v, k, λ, μ) and eigenvalues k, r, s with $s < -1$, which satisfies*

$$r > \tfrac{1}{2}s(s+1)(\mu+1) - 1$$

is the point graph of a dual linear space or dual transversal design.

The inequality reduces to Bose's in the pseudo-geometric case, but Neumaier's result applies without this assumption.

Theorem 4.6 *For any negative integer m, there is a finite list $\mathcal{L}(m)$ of strongly regular graphs with the property that, if G is any connected strongly regular graph whose adjacency matrix has eigenvalue m, then G is a complete multipartite graph with block size $-m$, or the point graph of a dual linear space or dual transversal design with $t+1 = -m$, or a member of the list $\mathcal{L}(m)$.*

For $m = -1$ the result is trivial, and for $m = -2$ it was proved by Seidel [25]: the list $\mathcal{L}(-2)$ contains only the Petersen, Clebsch, Schläfli, Shrikhande and three Chang graphs. However, all but twelve of the 32548 graphs with parameters $(36, 15, 6, 6)$ mentioned earlier belong to $\mathcal{L}(-3)$; that is, only twelve come from Latin squares.

5 Eigenvalues and their geometry

Let G be a strongly regular graph with vertex set $V = \{v_1, \ldots, v_n\}$ and adjacency matrix A. As we have seen, A has just three distinct eigenvalues

k, r, s, with multiplicities $1, f, g$ respectively (so that $1 + f + g = n$); the eigenvector associated to the eigenvalue k is the all-1 vector. Thus, $A = kE_0 + rE_1 + sE_2$, where E_0, E_1, E_2 are the orthogonal projections of $\mathbb{R}^n$ onto the three eigenspaces V_0, V_1, V_2 of A.

We fix attention on one of the non-trivial eigenspaces, say V_1, and consider the projections of the vertices (the basis vectors of $\mathbb{R}^n$) onto V_1. Thus, let $x_i = v_i E_1$ for $i = 1, \ldots, n$. The basic property of these vectors is the following:

Proposition 5.1 *There are real numbers α, β, γ (expressible in terms of the parameters of G) such that the inner products of the vectors x_i are given by*

$$\langle x_i, x_j \rangle = \begin{cases} \alpha, & \text{if } v_i = v_j; \\ \beta, & \text{if } v_i \sim v_j; \\ \gamma, & \text{if } v_i \neq v_j \text{ and } v_i \not\sim v_j. \end{cases}$$

Moreover, if G is connected and not complete multipartite, then $x_i \neq x_j$ for $i \neq j$.

In particular, if G is connected and not complete multipartite (as we will assume without comment for the rest of this section), then the vectors $x_1, \ldots, x_n$ lie on a sphere of radius $\sqrt{\alpha}$ in $\mathbb{R}^f$, and the angular distances between them take one of two possible values $\arccos \beta/\alpha$ (for adjacent vertices) and $\arccos \gamma/\alpha$ (for non-adjacent vertices). It is more convenient to re-scale the vectors by $1/\sqrt{\alpha}$ so that they lie on the unit sphere.

Delsarte *et al.* [11] proved:

Theorem 5.2 *The cardinality n of a two-distance set on the unit sphere in $\mathbb{R}^f$ satisfies*

$$n \leq \binom{f+2}{2} - 1.$$

This result can be translated into an inequality on the parameters of a strongly regular graph (which is connected and not complete multipartite). This is the so-called *absolute bound*. The same authors also gave a *special bound* depending on the values of α, β, γ (that is, on the actual distances realised by the set); it does not apply for all values of the parameters, but it is sometimes more powerful than the absolute bound.

A set $X = \{x_1, \ldots, x_n\}$ of vectors lying on the unit sphere $\Omega = S^{f-1}$ in Euclidean space $\mathbb{R}^f$ is called a *spherical t-design* if, for any polynomial function F of degree at most t, we have

$$\frac{1}{n}\sum_{i=1}^{n} F(x_i) = \frac{1}{\mathrm{vol}(\Omega)}\int_{\Omega} F(x)\,\mathrm{d}x.$$

In other words, the finite set "approximates the sphere up to degree t". For small t, there is a mechanical interpretation. Place unit masses at the points of X. Then X is a spherical 1-design if and only if the centre of mass is at the origin, and is a spherical 2-design if, in addition, the inertia ellipsoid is a sphere (that is, the moments of inertia are all equal and the products of inertia are zero).

Theorem 5.3 *Let G be a connected strongly regular graph which is not complete multipartite, and let X be the projection of the vertex set of G onto a non-trivial eigenspace, re-scaled to lie on the unit sphere. Then*

(a) X is a spherical 2-design;

(b) X is a spherical 3-design if and only if the Kreĭn bound corresponding to this eigenspace is attained (and this implies that G satisfies $C(3)$);

(c) X is a spherical 4-design if and only if the absolute bound is attained (and this implies that G satisfies $C(4)$);

(d) X is never a spherical 5-design.

We saw that the parameters of graph satisfying $C(3)$ (apart from the pentagon C_5) are either of Latin or negative Latin square type or of Smith type. Such a graph gives a spherical 3-design if and only if it attains the Kreĭn bound). All Smith graphs, and the pentagon, attain this bound, but no graphs of Latin square type do. For negative Latin square type $NL_r(n)$, a 3-design arises if and only if $n = r(r+3)$. Only two such graphs are known, the Clebsch graph on 16 vertices ($r = 1$) and the Higman–Sims graph on 100 vertices ($r = 2$).

We obtain a spherical 4-design in the smaller eigenspace if and only if the graph G is C_5 or a so-called "extremal Smith graph". Two examples of extremal Smith graphs are known, the Schläfli graph on 27 vertices and the McLaughlin graph on 275 vertices.

Information on the geometry of eigenspaces for more general classes of graphs is given in Chapter **??**; see also Godsil [15].

6 Rank 3 graphs

Looking again at the picture of the Petersen graph with which we began, we see that it has fivefold symmetry, and indeed has the symmetry of a regular pentagon (the dihedral group of order 10). In fact, there is more symmetry, not visible in the diagram. The graph has a well-known representation as the complement of the line graph of K_5; that is, the vertices can be labelled with the 2-element subsets of $\{1,\dots,5\}$, in such a way that two vertices are adjacent if and only if their labels are disjoint. Now the symmetric group S_5, in its induced action on the vertex labels, acts as a group of automorphisms of the graph. It is not hard to show that S_5 is the full automorphism group. Moreover, S_5 acts transitively on the set of adjacent pairs of vertices and on the set of non-adjacent pairs of vertices.

A graph G is called a *rank* 3 *graph* if it admits a group $\mathcal{G}$ of automorphisms with the property that $\mathcal{G}$ acts transitively on the set of vertices, on the set of ordered pairs of adjacent vertices, and on the set of ordered pairs of non-adjacent vertices. The term comes from permutation group theory, where the *rank* of a transitive permutation group $\mathcal{G}$ on Ω is the number of orbits of $\mathcal{G}$ on the set of ordered pairs of elements of Ω. In the case of a rank 3 graph, with $\Omega = VG$, the three orbits are $\{(v,v) : v \in VG\}$, $\{(v,w) : v \sim w\}$, and $\{(v,w) : v \neq w, v \not\sim w\}$.

Proposition 6.1 *(a) A rank* 3 *graph is strongly regular.*

(b) Let $\mathcal{G}$ be a permutation group which is transitive, has rank 3*, and has even order. Then there is a rank* 3 *graph G admitting $\mathcal{G}$ as a group of automorphisms.*

Proof (a) The number of neighbours of a vertex (or common neighbours of a pair of vertices) is clearly the same as the number of (common) neighbours of any image under an automorphism. The result follows.

(b) The group $\mathcal{G}$ has just two orbits on ordered pairs of *distinct* elements of Ω, say O_1 and O_2. Now, for any orbit O, the set $O^* = \{(w,v) : (v,w) \in O\}$ is also an orbit. So either $O_1^* = O_1$, or $O_1^* = O_2$. However, since $\mathcal{G}$ has even order, it contains an element g of order 2. This element interchanges two points v, w of Ω. If $(v,w) \in O_i$, then $O_i^* = O_i$. So the first alternative holds. Now take the graph G to have vertex set Ω, with $v \sim w$ whenever $(v,w) \in O_1$. Our argument shows that the graph is undirected; clearly it admits $\mathcal{G}$ as a rank 3 group of automorphisms. This is a special case of the general construction of G-invariant graphs in Chapter **??**.

Note that the orbits O_1 and O_2 give rise by this construction to complementary strongly regular graphs.

A major result in permutation group theory, which relies heavily on the Classification of Finite Simple Groups, is the determination of all the rank 3 permutation groups. We outline the argument. Let $\mathcal{G}$ be a rank 3 permutation group on Ω.

We call $\mathcal{G}$ *imprimitive* if it preserves a non-trivial equivalence relation, and *primitive* otherwise. Now, if G is imprimitive, let $\equiv$ be the equivalence relation preserved by $\mathcal{G}$; then the sets

$$\{(v, w) : v \equiv w, v \neq w\} \text{ and } \{(v, w) : v \not\equiv w\}$$

are $\mathcal{G}$-invariant, so must be the two $\mathcal{G}$-orbits on pairs of distinct points. The corresponding graphs are the disjoint union of complete graphs and the complete multipartite graph. So we may assume that $\mathcal{G}$ is primitive.

The basic analysis of such a group is done by considering the *socle* of $\mathcal{G}$, the product of its minimal normal subgroups. It follows from the O'Nan–Scott Theorem that one of three possibilities must occur for the socle $\mathcal{N}$ of $\mathcal{G}$ (see Chapter **??**):

(a) $\mathcal{N}$ is elementary abelian and acts regularly;

(b) $\mathcal{N}$ is a non-abelian simple group;

(c) $\mathcal{N}$ is the direct product of two isomorphic non-abelian simple groups.

In case (a), because its action is regular, $\mathcal{N}$ can be identified with the set of points permuted, and is the additive group of a vector space V over the field $\mathrm{GF}(p)$, for some prime p. The subgroup $\mathcal{H}$ fixing the origin is a group of linear transformations of V, with two orbits X_1 and X_2. In our case, the orbits satisfy $X_1 = -X_1$ and $X_2 = -X_2$, and the complementary graphs G_1 and G_2 have vertex set V and satisfy $v \sim w$ in G_i if and only if $v - w \in X_i$.

So the classification in this case is reduced to finding groups of matrices over $\mathrm{GF}(p)$ having just two orbits (each closed under negation) on non-zero vectors. Examples include:

- The multiplicative group of the non-zero squares in GF(q), where $q \equiv 1 \pmod 4$. The orbits are the sets of squares and non-squares in GF(q), and the graphs (which happen to be isomorphic) are the Paley graph $P(q)$.

- The *orthogonal group* preserving a non-degenerate quadratic form over GF(2); the orbits are the sets of non-zero vectors v satisfying $Q(v) = \alpha$, for $\alpha = 0, 1$. Such forms can be defined on spaces of even dimension, and there are just two inequivalent forms. For example, in dimension 4, the quadratic forms

$$x_1x_2 + x_3x_4 \text{ and } x_1x_2 + x_3^2 + x_3x_4 + x_4^2$$

  give the graphs $L_2(4)$ (and its complement) and the *Clebsch graph* (and its complement) respectively. In general, these graphs occur among Thomason's pseudo-random graphs [28]. They are of pseudo or negative Latin square type, and satisfy $C(3)$.

The complete list of linear groups with two orbits on non-zero vectors was determined by Liebeck [19].

In case (b), where the socle $\mathcal{N}$ of $\mathcal{G}$ is non-abelian simple, the Classification of Finite Simple Groups shows that it must be an alternating group, a group of Lie type, or one of the twenty-six sporadic groups. Moreover, the O'Nan–Scott Theorem gives the extra information that $\mathcal{G}$ lies between $\mathcal{N}$ and its automorphism group (where $\mathcal{N}$ is embedded in Aut($\mathcal{N}$) as the group of inner automorphisms). (A group $\mathcal{G}$ lying between a simple group and its automorphism group is said to be *almost simple*.)

The combined efforts of a number of mathematicians including Bannai, Kantor, Liebler, Liebeck, and Saxl, have determined all rank 3 actions of almost simple groups. Examples include:

- The symmetric group $\mathcal{S}_n$ (for $n \geq 5$), acting on the set of 2-element subsets of $\{1, \ldots, n\}$; this gives the *triangular graph* $T(n)$ and its complement.

- The *projective group* PGL(n, q) (for $n \geq 4$) has a rank 3 action on the set of lines of the projective space: the orbits are the sets of intersecting pairs and skew pairs of lines.

- A *classical group* (one preserving a polarity of a projective space) acts on the set of self-polar points of the projective space (these form the *polar space* associated with the polarity; the action has rank 3 except in a few low-dimensional cases where it is doubly transitive. For a few cases involving small fields, the action on the non-self-polar points also has rank 3.

- There are various "sporadic" examples as well, such as $\mathrm{PSU}(3, 5^2)$ on the vertices of the Hoffman–Singleton graph, or the Higman–Sims group on the vertices of the Higman–Sims graph.

Several of the sporadic simple groups were first constructed as groups of automorphisms of strongly regular graphs. These were the Hall–Janko, Higman–Sims, McLaughlin, Suzuki, Fischer and Rudvalis groups.

In case (c) of the O'Nan–Scott Theorem, the socle $\mathcal{N}$ of $\mathcal{G}$ is the direct product of two isomorphic simple groups. The analysis leading to this case actually shows that the rank 3 graphs which arise are the lattice graphs $L_2(n)$ and their complements.

7 Related classes of graphs

There are many generalisations or variants of strongly regular graphs. In this section we introduce a few of these: distance-regular graphs, association schemes, walk-regular graphs, edge-regular graphs, Deza graphs, and strong graphs.

A connected graph G of diameter d is *distance-regular* if there are constants c_i, a_i, b_i for $0 \le i \le d$ such that, if u and v are vertices at distance i, then the number of vertices w such that $w \sim v$ and w is at distance $i-1, i, i+1$ from u is c_i, a_i, b_i respectively. The numbers c_i, a_i, b_i are the *parameters* of the graph. Note that c_0, a_0 and b_d are zero.

A distance-regular graph is regular; its valency is $b_0 = k$. We have $c_i + a_i + b_i = k$ for all i, and $c_1 = 1$. Thus there are $2d - 3$ "independent" parameters. A distance-regular graph of diameter 2 is the same thing as a connected strongly regular graph; we have $\lambda = a_1$ and $\mu = c_2$.

A connected graph G is *distance-transitive* if there is a group $\mathcal{G}$ of automorphisms of G such that, for any two pairs (u_1, v_1) and (u_2, v_2) of vertices satisfying $d(u_1, v_1) = d(u_2, v_2)$, there is an automorphism $g \in \mathcal{G}$ which maps

u_1 to u_2 and v_1 to v_2. It is clear that any distance-transitive graph is distance-regular, and that a distance-transitive graph of diameter 2 is the same thing as a connected rank 3 graph.

The determination of all distance-transitive graphs is not yet complete. This class of graphs is discussed further in Chapter **??**. Further information about distance-regular and distance-transitive graphs is contained in the book by Brouwer, Cohen and Neumaier [6].

There are many distance-regular graphs which are not distance-transitive. Such graphs become less common as the diameter increases. However, as shown by Egawa [12], there exist distance-regular graphs of arbitrarily large diameter which are not distance-transitive.

The adjacency matrix of a regular connected graph of diameter d has at lest $d+1$ distinct eigenvalues (one of which is the valency). Distance-regular graphs attain this bound. As we have seen, a regular connected graph has three eigenvalues if and only if it is strongly regular. However, for $d \geq 3$, there are regular connected graphs of diameter d with d eigenvalues which are not distance-regular. The first examples were constructed by Bridges and Mena [4]; but the study of this interesting class of graphs has not yet progressed much beyond the collection of examples.

Looking more closely at distance-regular graphs, it can be shown that there are constants p_{ij}^k for $0 \leq i, j, k \leq d$ with the property that, given vertices u and v with $d(u, v) = k$, the number of vertices w such that $d(u, w) = i$ and $d(w, v) = j$ is precisely p_{ij}^k. Now we can generalise this as follows.

Suppose that the ordered pairs of points of a set Ω are partitioned into $s + 1$ classes $C_0, \ldots, C_s$ with the following properties:

- The diagonal $\{(v, v) : v \in \Omega\}$ is a single class C_0;
- Each class C_i is symmetric (that is, if $(u, v) \in C_i$ then $(v, u) \in C_i$);
- Given $i, j, k \in \{1, \ldots, s\}$ and $(u, v) \in C_i$, the number of w such that $(u, w) \in V_j$ and $(w, v) \in C_k$ depends only on i, j, k and not on (u, v).

Such a structure is called an *association scheme*. Thus, a distance-regular graph gives rise to an association scheme.

More about association schemes can be found in Bannai and Ito [2], Godsil [15] and Bailey [1]. A still more general concept is a *coherent configuration*, where the relations are not required to be symmetric.

Association schemes were originally used in experimental design by Bose and his school. Suppose that an experiment is being performed on a number of experimental units which are divided into b blocks of size k (for example, k fields on each of b farms, or k patients in each of b hospitals). We want to apply a number v of different treatments in such a way that no treatment occurs more than once in the same block. It is clearly a good idea to arrange that any two treatments occur together in a block the same number of times, if possible. (Such a design is called *balanced*.) However, Fisher showed that, if $k < v$, this is not possible unless $v \leq b$. To test more treatments,we must relax the condition of balance. Bose observed that the best approach is to have an association scheme on the set of treatments, and to arrange that the number of times two treatments u and v occur together in a block depends only on which associate class C_i contains (u, v). Such a design is called *partially balanced*. See [1] for more information.

Indeed, this is a case where the applications came first, and the generalisation preceded the special case. Partially balanced designs were defined by Bose and Nair in 1939. During the 1950s, association schemes became of interest in their own right, but not until Bose's 1963 paper [3] was the term "strongly regular graph" introduced.

A graph G is *walk-regular* if, for every non-negative integer i and vertex v, the number of closed walks of length i starting at v depends only on i, not on v. Equivalently, a graph is walk-regular if the characteristic polynomials of all its vertex-deleted subgraphs are the same. The class of walk-regular graphs includes both the vertex-transitive graphs and the distance-regular graphs, and of course is contained in the class of regular graphs. See Godsil [15] for more about these graphs.

A strongly regular graph is defined by three conditions:

(a) any vertex has k neighbours;

(b) any two adjacent vertices have λ common neighbours;

(c) any two non-adjacent vertices have μ common neighbours.

We can weaken the definition by requiring only two of the three conditions to hold. A graph satisfying (a) and (b) is called *edge-regular*; a graph satisfying (b) and (c) is a *Deza graph* [13]. The class of graphs satisfying (a) and (c) has not been studied except in special cases.

More systematically, recall that a graph is t-tuple regular (that is, satisfies $C(t)$) if the number of common neighbours of a set S of at most t vertices depends only on the isomorphism type of the induced subgraph on S. Let us say that a graph satisfies $R(t)$ if this condition holds for sets S with $|S| = t$. Thus, a Deza graph satisfies $R(2)$ but not necessarily $R(1)$. As far as I know, no systematic study of the possible sets of integers t for which $R(t)$ can hold in a graph has been made.

A variant of Deza graphs was earlier introduced by Seidel, who defined a *strong graph* to be one having the property that, for any two vertices u and v, the number of vertices joined to just one of the two depends only on whether or not u and v are joined. Using a modified adjacency matrix B with 0 on the diagonal, -1 for adjacency and $+1$ for non-adjacency, we find that

$$(B - \rho_1 I)(B - \rho_2 I) = (n - 1 + \rho_1\rho_2)J$$

for some ρ_1, ρ_2. It follows that, if $n-1+\rho_1\rho_2 \neq 0$, the graph is regular (and so strongly regular). In the remaining case, when $n-1+\rho_1\rho_2 = 0$, the graph need not be regular; such *special* strong graphs are closely connected with regular two-grahs (see below).

The operation σ_X of *switching* a graph G with respect to a set X of vertices is defined as follows: edges between X and its complement are "switched" to non-edges, and non-edges to edges; adjacencies within X or outside X remain unaltered. Switching with respect to all subsets generates an equivalence relation on the class of all graphs on a fixed vertex set V. It is easy to see that, if $\mathcal{T}$ is the set of 3-subsets of V which contain an odd number of edges of G, then $\mathcal{T}$ is unaltered by switching. Moreover, a set $\mathcal{T}$ of triples arises from a graph in this way if and only if any 4-set contains an even number of members of $\mathcal{T}$. Such a set is called a *two-graph.* Thus, there is a bijection between the set of two-graphs on V and the set of switching equivalence classes on V.

Switching a graph G has the effect of pre- and post-multiplying the $(0, -1, +1)$ adjacency matrix of G (defined above) by a diagonal matrix with entries ± 1. The matrix equation

$$(B - \rho_1 I)(B - \rho_2 I) = 0$$

satisfied by special strong graphs is unaffected by this, so if a graph satisfies this equation, so do all graphs in its switching class. In this case the corresponding two-graph is called *regular.* Regular two-graphs are also characterised by the property that any two vertices in V lie in a constant number of triples in $\mathcal{T}$.

There are many connections between regular two-graphs, strongly regular graphs, sets of equiangular lines in Euclidean space, doubly transitive permutation groups, antipodal distance-regular graphs of diameter 3, and several other topics. We refer to Seidel's surveys [26, 27].

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