A generalisation of t-designs

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Abstract

This paper defines a class of designs which generalise t-designs, resolvable designs, and orthogonal arrays. For the parameters $t = 2$, $k = 3$ and $\lambda = 1$, the designs in the class consist of Steiner triple systems, Latin squares, and 1-factorisations of complete graphs. For other values of t and k, we obtain t-designs, Kirkman systems, large sets of Steiner triple systems, sets of mutually orthogonal Latin squares, and (with a further generalisation) resolvable 2-designs and indeed much more general partitions of designs, as well as orthogonal arrays over variable-length alphabets.

The Markov chain method of Jacobson and Matthews for choosing a random Latin square extends naturally to Steiner triple systems and 1-factorisations of complete graphs, and indeed to all designs in our class with $t = 2$, $k = 3$, and arbitrary λ, although little is known about its convergence or even its connectedness.

1 Introduction

This paper starts from the observation that there are striking similarities among Steiner triple systems, Latin squares, and 1-factorisations of complete graphs. For all three cases, the number of designs of admissible order grows rapidly (the logarithm of the counting function is asymptotically $cn^2 \log n$), and almost all of these designs admit no non-trivial automorphisms. There are similar results about subdesigns (we prove such a result for 1-factorisations in this paper). Moreover, as we will see, very similar Markov chains can be defined for all three types.

We define a concept which, in the case $t = 2$, $k = 3$, $\lambda = 1$, gives precisely these three types of design, but which has generalisations to arbitrary t-designs, resolvable designs, and orthogonal arrays.

We use the notation $\binom{X}{k}$ to denote the set of all k-subsets of the set X.

2 Definition and basic properties

Let t, k, λ be given positive integers with $\lambda > 0$ and $k > t > 0$ Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a composition of k (that is, a tuple of positive integers with sum k), and let $\mathbf{v} = (v_1, \ldots, v_m)$, where $v_i \geq k_i$ for all i. Then we define a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design to consist of an m-tuple $\mathbf{X} = (X_1, \ldots, X_m)$ of pairwise disjoint sets with $|X_i| = v_i$ for $i = 1, \ldots, m$, and a set

$$\mathcal{B} \subseteq \binom{X_1}{k_1} \times \cdots \times \binom{X_m}{k_m}$$

with the following property:

> if $\mathbf{t} = (t_1, \ldots, t_m)$ is a t-tuple of integers with sum t satisfying $0 \leq t_i \leq k_i$ for $i = 1, \ldots, m$, then for any choice $\mathbf{T} = (T_1, \ldots, T_m)$ with $T_i \in \binom{X_i}{t_i}$ for $i = 1, \ldots, m$, there are precisely λ members $\mathbf{K} = (K_1, \ldots, K_m) \in \mathcal{B}$ for which $T_i \subseteq K_i$ for $i = 1, \ldots, m$.

The *order* of the design is the m-tuple $\mathbf{v}$.

Note that, in the case when $\mathbf{k} = (k)$, $\mathbf{v} = (v)$, this is precisely the definition of a t-(v, k, λ) design.

There are some necessary conditions on the order which must be satisfied in order for a design to exist.

Proposition 1 *Suppose that a* t-$(\mathbf{v}, \mathbf{k}, \lambda)$ *design* $\mathcal{B}$ *exists with* $\mathbf{k} = (k_1, \ldots, k_m)$ *and order* $\mathbf{v} = (v_1, \ldots, v_m)$*, with* $|\mathcal{B}| = b$*. Then, for any* m*-tuple* $\mathbf{t} = (t_1, \ldots, t_m)$ *summing to* t *with* $0 \leq t_i \leq k_i$ *for* $i = 1, \ldots, m$*, we have*

$$b \prod_{i=1}^{m} \binom{k_i}{t_i} = \lambda \prod_{i=1}^{m} \binom{v_i}{t_i}.$$

The proof is straightforward counting.

This theorem gives necessary conditions for the existence of a design. As well as divisibility conditions (as we expect from the case of t-designs), sometimes it follows directly from the parameters that no non-trivial designs can exist.

Here is an example. Let $t = 2$, $k = 4$, $\mathbf{k} = (2, 2)$, and $\mathbf{v} = (v_1, v_2)$. Then we have

$$\begin{aligned} 4b &= \lambda v_1 v_2, \\ b &= \lambda \binom{v_1}{2} = \lambda \binom{v_2}{2} \end{aligned}$$

giving equations $v_1 = v_2$, $v_1 = 2(v_2 - 1)$, $v_2 = 2(v_1 - 1)$, solvable only in the trivial case $v_1 = v_2 = 2$.

In fact, it is straightforward to check that, for $t = 2$ and $k = 4$, the only partitions $\mathbf{k}$ allowing non-trivial designs are (4), $(1, 1, 1, 1)$, and $(3, 1)$. It seems that this phenomenon occurs for $1 < t < k - 1$, but not for $t = 1$ or $t = k - 1$ (see below).

Proposition 2 *Suppose that a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design exists with $\mathbf{k} = (k_1, \ldots, k_m)$ and order $\mathbf{v} = (v_1, \ldots, v_m)$. Suppose that the t-tuple $\mathbf{t} = (t_1, \ldots, t_m)$ has sum t and satisfies $0 \le t_i \le k_i$ for $i = 1, \ldots, m$. Suppose also that, for fixed $p \ne q$, we have $t_p > 0$ and $t_q < k_j$. Then*

$$(k_p - t_p + 1)(v_q - t_q) = (k_q - t_q)(v_p - t_p + 1).$$

Proof Put $t'_p = t_p - 1$, $t'_q = t_q + 1$, and $t'_i = t_i$ for $i \ne p, q$. The numbers $t'_1, \ldots, t'_m$ also sum to t and satisfy $0 \le t'_i \le k_i$ for all i. So we can apply the previous proposition to obtain two expressions for b. Equating them gives the result.

Corollary 3 *A necessary condition for the existence of a 1-$(\mathbf{v}, \mathbf{k}, \lambda)$ design is that v_i/k_i is constant for $i = 1, \ldots, m$. Moreover, if $v_i = rk_i$ for $1 \le i \le m$, for a fixed integer r, then such a design exists.*

Proof The only way of applying the proposition is with $t_p = 1$ and $t_q = 0$, when it gives $k_p v_q = v_p k_q$, giving the result.

Now if $v_i = rk_i$ for all i, choose a partition of X_i into r sets of size k_i, say $(Y_{i1}, \ldots, Y_{ir})$, for each i; let

$$\mathcal{B} = \{(Y_{1j}, \ldots, Y_{mj}) : j = 1, \ldots, m\}.$$

It is easily verified that this is a 1-$(\mathbf{v}, \mathbf{k}, 1)$ design. Repeating the "blocks" λ times gives the general case.

Corollary 4 *Suppose that a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design exists with $k = t + 1$. Then $v_i - k_i$ is constant for $i = 1, \ldots, m$.*

Proof Apply the proposition with $t_p = k_p$ and $t_q = k_q - 1$.

3 Small k and t

As we have seen, the case $t = 1$ is uninteresting. We consider the next few cases.

3.1 The case $t = 2$, $k = 3$

In this case we obtain precisely our motivating examples.

Case $\mathbf{k} = (3)$ We have a collection of 3-subsets of $X = X_1$ such that any two points of X lie in exactly λ of them; that is, a 2-$(v, 3, \lambda)$ design, where $v = |X|$. For $\lambda = 1$, this is a Steiner triple system.

Case $\mathbf{k} = (1, 1, 1)$ We have a collection $\mathcal{B}$ of elements of $X_1 \times X_2 \times X_3$ such that, if we pick any two of the three sets X_1, X_2, X_3 and an element from each set, there are exactly λ elements of $\mathcal{B}$ having those entries in the appropriate positions; an orthogonal array of strength 2 and degree 3. For $\lambda = 1$, this is equivalent to a Latin square, where X_1, X_2, X_3 are the sets of rows, columns, entries.

Case $\mathbf{k} = (2, 1)$ In the case $\lambda = 1$, this is equivalent to a 1-factorisation of the complete graph on X_1, with X_2 a set indexing the 1-factors. An element of $\mathcal{B}$ consists of an edge and the index of the 1-factor containing it. Two distinct points of X_1 are together in a unique 1-factor, and given any point and any 1-factor, there is a unique pair in the 1-factor containing the point. (For arbitrary λ, we have a λ-factorisation of the complete multigraph with edge-multiplicity λ; that is, each factor is a (multi)graph with degree λ.)

3.2 The case $t = 2$, $k = 4$

We saw earlier that there are no non-trivial designs when $\mathbf{k} = (2, 2)$, and the same is true for $\mathbf{k} = (2, 1, 1)$. For simplicity we take $\lambda = 1$.

Case $\mathbf{k} = (4)$ Here we have Steiner systems $S(2, 4, v)$.

Case k $= (3,1)$ We have $v_2 = (v_1 - 1)/2$. A block consists of a 3-subset of X_1 together with a point of X_2 which we can regard as a label. The 3-sets which arise form a Steiner triple system, and the 3-sets labelled by a fixed element of X_2 form a 1-factor. So the design is a *Kirkman system*, a resolved Steiner triple system (a Steiner triple system with a specified resolution).

Case k $= (1,1,1,1)$ We have $v_1 = v_2 = v_3 = v_4$, and $\mathcal{B}$ is an orthogonal array of strength 2 and degree 4, equivalent to a pair of orthogonal Latin squares.

3.3 The case $t = 3$, $k = 4$

Again we assume that $\lambda = 1$ for simplicity.

Case k $= (4)$ Here we have Steiner quadruple systems $S(3,4,v)$.

Case k $= (3,1)$ We have $v_2 = v_1 - 2$. A block consists of three points of X_1 labelled by a point of X_2. Each 3-subset of X_1 occurs exactly once, and the points with a given label form a Steiner triple system. So the design is a *large set* of Steiner triple systems, a partition of $\binom{X_1}{3}$ into Steiner triple systems. Such sets exist for all admissible orders greater than 7.

Case k $= (2,2)$ Now $X_1 = X_2$. A design of this type is described by a function f from $\binom{X_1}{2}$ to the set of 1-factors on X_2, and a function g from $\binom{X_2}{2}$ to the set of 1-factors on X_1, such that, if P_i is a 2-subset of X_i for $i = 1,2$, then $P_2 \in f(P_1)$ if and only if $P_1 \in g(P_2)$. Here are three entirely different types of example.

- Take two 1-factorisations on X_1 and X_2, two sets of the same size v, with a bijection between the 1-factors. Now (P_1, P_2) is a block if P_i is a 2-subset of X_i for $i = 1,2$ and the 1-factors containing P_1 and P_2 correspond.

- Suppose that there exists a Steiner quadruple system $S(3,4,2v)$ containing a subsystem $S(3,4,v)$. (This requires $v \equiv 2$ or 4 mod 6.) The complement of such a subsystem is also a subsystem. Now let X_1 and X_2 be the subsystem and its complement. Let $\mathcal{B}$ consist of all pairs

(P_1, P_2) for which P_i is a 2-element subset of X_i such that $P_1 \cup P_2$ is a block of the SQS.

- A special case of both the above is obtained by taking X_1 and X_2 to be the same elementary abelian group of order 2^d for some d, and letting $\mathcal{B}$ consist of all pairs $(\{x_1, y_1\}, \{x_2, y_2\})$ for which $x_1 + y_1 = x_2 + y_2 \neq 0$.

- The remarkable outer automorphism of S_6 gives another example. Let X_1 be a set of six points, and X_2 the set of six 1-factorisations of the complete graph on X. Then $\mathcal{B}$ consists of pairs (P_1, P_2), where the 2-set P_1 belongs to the unique 1-factor common to the two 1-factorisations in P_2.

Case $\mathbf{k} = (2, 1, 1)$ Here is a family of examples. (There are others.) Take a 1-factorisation on a set of size v, with factors $f_1, \ldots, f_{v-1}$, and a Latin square L of order $v - 1$. Now the blocks have the form $(\{x, y\}, i, j)$, where $\{x, y\} \in f_k$ and the (i, j) entry of L is k.

Case $\mathbf{k} = (1, 1, 1, 1)$ The design is an orthogonal array of strength 3 and index 4, equivalent to a Latin cube. Lots of these exist. For example, take two Latin squares of order v with the same symbol set, say L_1 and L_2, and consider all quadruples (i, j, k, l) such that $(L_1)_{ij} = (L_2)_{kl}$. Alternatively, take a group G of order v and consider all quadruples (i, j, k, l) for which $g_i g_j g_k g_l = 1$.

4 Triple systems

As explained, the three types of design with $t = 2$, $k = 3$, and $\lambda = 1$ have many common features. Often a theorem which has been proved for one class can be extended to the others. We can also ask whether such results can be extended to other values of t, k and λ.

We give here a brief survey of some examples. Recall that $\mathbf{k} = (3)$ for Steiner triple systems, $(2, 1)$ for 1-factorisations, and $(1, 1, 1)$ for Latin squares. The natural number n is *admissible* in the first case if $n \equiv 1$ or $3 \pmod 6$; in the second if n is even; and in the third, every natural number is admissible.

The number of designs of admissible order n. In each case, the number $F(n)$ satisfies $\log F(n) \sim cn^2 \log n$. We have $c = \frac{1}{6}$, $\frac{1}{2}$ and 1 respectively in the three cases. The proofs are based on the van der Waerden permanent conjecture. More accurate estimates are known in the first and third case (see [14, 17]).

Almost all have trivial automorphism group. This was proved for $\mathbf{k} = (3)$ by Babai [1], and in the other two cases by the author (unpublished) at about the same time; a proof is to be published soon [15].

Block-transitive designs. The Steiner triple systems which have block-transitive automorphism groups are known: they are projective spaces over GF(2), affine spaces over GF(3) and Netto systems (Clapham [7]). Such a characterisation is not known in the other cases, and it is unlikely that one will be found. For example, the Cayley table of any group is a Latin square with block-transitive automorphism group.

However, if one imposes a stronger condition, a uniform classification is known. The number of possible relations between two blocks is $m+2$, where m is the number of sets X_i: they may be equal, they may intersect in a point of X_i for $i = 1, \ldots, n$, or they may be disjoint. (This assumes that the order is large enough that disjoint blocks exist.) Now designs whose automorphism group has just $m+2$ orbits on pairs of blocks are all known:

- for Steiner triple systems, we have just projective spaces over GF(2) and the affine plane of order 3 with 9 points (Higman [10]);
- for 1-factorisations, we have affine spaces over GF(2) and a unique example on 6 points (this is unpublished as far as I know, but is an exercise for the reader);
- for Latin squares, we have the Cayley tables for elementary abelian 2-groups and the cyclic group of order 3 (Bailey [2]).

All the proofs are "elementary" (not relying on the Classification of Finite Simple Groups).

Embeddings of designs. A design of admissible order n is embeddable in designs of all admissible orders at least $f(n)$, where $f(n) = 2n+1$, $2n$, $2n$ in the three cases respectively. This is easy for Latin squares. For suppose that $m \geq 2n$. Extend the Latin square of order n to a $n \times m$ Latin rectangle using $m-n$ new symbols; then extend this Latin rectangle to a Latin square. For Steiner triple systems it is a result of Doyen and Wilson [8]. I do not know a proof in the literature for 1-factorisations (this is mentioned as an open problem in [4]), so one is given below.

Embeddings of partial designs. A partial design of order n is embeddable in designs of all admissible orders at least $f(n)$, with the same function f as in the last paragraph. This is a theorem of Ryser [13] for Latin squares, and of Bryant and Horsley [3] for Steiner triple systems. I do not know of a proof for 1-factorisations: this is an open problem. Of course, one has to be more careful about what constitutes a partial design in this case; we assume that $|X_1| = n$ and $|X_2| = n-1$, and that $\mathcal{B}$ is a set of triples each containing two elements of X_1 and one of X_2, so that any two points of X_1, or any 2-set containing one point from each of X_1 and X_2, is contained in at most one block.

Embedding partial designs so that automorphisms extend. There is a function $f(n)$, growing exponentially with n, such that a partial Steiner triple system P of order n is embeddable in a Steiner triple system S of every admissible order at least $f(n)$ such that all automorphisms of P are induced by automorphisms of S (Cameron [6]). Such a result is not known in the other two cases but is presumably not too difficult.

Every group is the automorphism group of a design. This is a result of Mendelsohn [12] for Steiner triple systems, and can be transferred to the other classes.

A Markov chain whose limiting distribution should be uniform. This is due to Jacobson and Matthews [11] for Latin squares. A generalisation to all types, and arbitrary values of λ, is proposed in this paper. Unfortunately, it is not known whether the Markov chain is connected in the other cases; if it is, then its limiting distribution will be uniform. This is another open problem.

Resolutions. A resolution class and a resolution are just the usual concepts for a Steiner triple system. In the case of a Latin square, a resolution class is a transversal (a set of v cells with one in each row, one in each column, and one carrying each symbol), while a resolution is a Latin square orthogonal to the given square.

This pattern does not complete: for $t = 2$, $\mathbf{k} = (2, 1)$ and $\lambda = 1$, only the trivial design has a resolution class. For each block contains two elements of X_1 and one of X_2, where $|X_1| = v$ and $|X_2| = v - 1$; so the existence of a resolution class would imply that $2(v - 1) = v$, whence $v = 2$.

4.1 Subdesigns

Proposition 5 *Suppose that m and n are even and $n > m$. Then there exists a 1-factorisation of order n with a subdesign of order m if and only if $n \geq 2m$.*

Proof The necessity of evenness of m and n is clear; for the inequality, suppose that a set X of size m in a 1-factorisation of order n carries a subdesign. Choose a point $y \notin X$. Then the edges $\{x, y\}$, for $x \in X$, all lie in distinct 1-factors not appearing within X; so $m + (m - 1) \leq n - 1$.

For the sufficiency, we simplify notation by putting $n = 2a$ and $m = 2b$. Put $a = b + r = 2b + t$; then $r \geq b$, so $t = r - b \geq 0$.

What we need for this construction is a complementary pair G_1 and G_2 of graphs on a set Y of $2r$ vertices, such that

- G_2 has degree $2b - 1$ and has a 1-factorisation,
- G_1 has degree $2t$ and its edge-set can be partitioned into $2r$ partial 1-factors $f_0, \ldots, f_{2r-1}$ each of size t,
- the set X_i of vertices not on edges of f_i can be written as $y_{i,0}, \ldots, y_{i,2b-1}$ such that, for fixed $j \in \{0, \ldots, 2b - 1\}$, each vertex in Y occurs as $y_{i,j}$ for a unique value of i.

Given such a structure, the construction is as follows. Given any 1-factorisation on a set $X = \{x_0, \ldots, x_{2b-1}\}$ of size $2b$, with factors $e_1, \ldots, e_{2b-1}$, we take the following 1-factors on $X \cup Y$:

- $e_i \cup g_i$ for $1 \leq i \leq 2b - 1$, where g_i is the ith factor of the graph G_2;
- for $0 \leq i \leq 2r - 1$, the set f_i together with the edges $\{x_j, y_{i,j}\}$ for $j \in \{0, \ldots, 2b - 1\}$.

This is clearly a 1-factorisation on $n = 2b + 2r$ vertices containing the given 1-factorisation on $m = 2b$ vertices.

We construct the required structure using a special row-complete Latin square of order $2r$ called a *Williams square* [16]. A Latin square is row-complete if each ordered pair of distinct symbols occurs exactly once in adjacent positions in a row of the square. The Williams square has symbol set and index sets of rows and columns as $\{0, \ldots, 2r - 1\}$; the 0th row is

$$(0, 1, 2r - 1, 2, 2r - 2, \ldots, r),$$

and the ith row is obtained by adding i mod $2r$. Let $L = (l_{i,j})$ denote this square.

Now we separate two cases. Suppose first that $2t \leq r$, that is, $n \leq 3m$. We partition the edges of the complete graph K_{2r} on the vertex set $Y = \{y_i : i \in \{0, \ldots, 2r - 1\}$ into two subgraphs G_1 and G_2 as follows. The first graph G_1 has $2rt$ edges, indexed by $\{l_{i,2b+2j}, l_{i,2b+2j+1}\}$ for $i = 0, \ldots, 2r - 1$ and $j = 0, 1, \ldots, t-1\}$. (That is, we take the last t consecutive pairs in each row.) By construction, $l_{i,2b+2j+1} - l_{i,2b+2j} = 2b + 2j + 1$; since we are in the case where $t < b$, this difference is in the interval $(r, 2r)$, and so its negative does not occur. Thus all the edges are distinct. Moreover, by the properties of a Latin square, G_1 is a regular graph of degree $2t$.

We have to show that the complementary graph G_2 has a 1-factorisation. Partition its vertex set into two parts A and B, each of size r, according to the parity of the index. Since all edges of G_1 join edges of opposite parity, G_2 consists of coomplete graphs on A and B together with a regular bipartite graph between these sets. Choose a 1-factorisation of the latter.

If r is even, we can take a 1-factorisation of each of the complete graphs and match up the 1-factors; the unions of pairs of 1-factors provide the remaining 1-factors of G_2.

If r is odd, take two dummy vertices α and β, and construct 1-factorisations of the complete graphs on $A \cup \{\alpha\}$ and $B \cup \{\beta\}$. Pick a 1-factor F between A and B. Now, for each edge $\{a, b\}$ of F, match the 1-factors containing $\{a, \alpha\}$ and $\{b, \beta\}$, delete these two edges and add the edge $\{a, b\}$ instead. The resulting 1-factors, together with the 1-factors other than F between A and B, form the required 1-factorisation of G_2.

In the other case, where $2t > r$, we proceed differently. The edges of G_1 are $\{l_{i,2j-1}, l_{i,2j}\}$ for $1 \leq j \leq \lfloor r/2 \rfloor$, together with edges $\{l_{i,2b+2j}, l_{i,2b+2j+1}\}$ for $0 \leq i \leq t - \lfloor r/2 \rfloor$. In other words, we take the first $\lfloor r/2 \rfloor$ pairs in each row (skipping the first element), and then some pairs from the end of the row. Again, the partial 1-factors are given by fixing the value of i. The edges of the first type have even differences and those of the second type have odd differences, so there is no overlap. Now G_1 consists of all the edges joining vertices of the same parity (if r is odd) or all except those joining antipodal points (if r is even), together with a regular subgraph of the complete bipartite graph between these two sets. So G_2 is regular bipartite, together with (in the case where r is even) one further 1-factor consisting of antipodal pairs; so G_2 has a 1-factorisation.

5 Markov chains for $t = 2$ and $k = 3$

We now present a Markov chain method of choosing a random 2-$(\mathbf{v}, \mathbf{k}, \lambda)$ design in the case when $k = 3$. This is a straightforward generalisation of the method given by Jacobson and Matthews [11] for Latin squares. Unfortunately, we cannot even prove that it is connected in general; if it is, then the limiting distribution is uniform. Note that other methods have been proposed in special cases (for example, hill-climbing for Steiner triple systems [9]); the present method has the theoretical advantage that (modulo the conjecture about connectedness) its limiting distribution is known to be uniform.

We denote by $[x, y, z]$ a triple of the appropriate form: for $\mathbf{k} = (1, 1, 1)$, this is an ordered triple from $X_1 \times X_2 \times X_3$; for $\mathbf{k} = (3)$, a 3-element subset of X_1 (so the triple is unordered); and for $\mathbf{k} = (2, 1)$, it has the form $(\{x, y\}, z)$ where $\{x, y\}$ is a 2-element subset of X_1 and $z \in X_2$. Let $X^{[3]}$ denote the set of all such triples. By $\lambda X^{[3]}$ we mean the multiset in which each triple occurs with multiplicity λ.

Now a t-$(\mathbf{v}, \mathbf{k}, \lambda)$ design can be regarded either as a multiset B of elements of $X^{[3]}$, or as a function f from $X^{[3]}$ to the non-negative integers such that

$$\sum_z f([x, y, z]) = \lambda$$

and similar equations for sums over x and y, where the summation variables range over the appropriate sets. Let $\mathcal{P}$ be the set of such functions, which we call *proper*.

An *improper* function is defined to be a function on $X^{[3]}$, satisfying the same summation conditions as above, and taking non-negative integer values except for the value -1 which is taken just once. Let $\mathcal{I}$ denote the set of all improper functions.

The basic Markov chain has state space $\mathcal{P} \cup \mathcal{I}$. It is defined as follows. Let f be a function in $\mathcal{P} \cup \mathcal{I}$.

1. Suppose that $f \in \mathcal{P}$. Choose a triple $[x, y, z]$ randomly and uniformly from $\lambda X^{[3]} \setminus B$. (This means that the probability of choosing a given triple is proportional to $\lambda - f([x, y, z])$). Go to Step 3.

   Note that the cardinality (which we will denote by N) of $\lambda X^{[3]} \setminus B$ is determined by the type and parameters: it is

$$N = \begin{cases} \lambda n^2(n-1) & \text{if } \mathbf{k} = (1,1,1), \\ \lambda n(n-1)(n-2)/2 & \text{if } \mathbf{k} = (2,1), \\ \lambda n(n-1)(n-3)/6 & \text{if } \mathbf{k} = (3). \end{cases}$$

2. Suppose that $f \in \mathcal{I}$. Choose $[x, y, z]$ to be the unique triple on which f takes the value -1.

3. Choose x' randomly, with probability proportional to $f([x', y, z])$ (excluding the case $x' = x$ in the improper case). Choose y', z' similarly.

4. If x', y', z', x, y, z are not all distinct, then return f.

5. Otherwise, increase the values of $f([x, y, z])$, $f([x', y', z])$, $f([x', y, z'])$ and $f([x, y', z'])$ by one; decrease the values of $f([x', y, z])$, $f([x, y', z])$, $f([x, y, z'])$ and $f([x', y', z'])$ by one; and return the resulting function (which is proper or improper according as the original value of $f([x', y', z'])$ was positive or zero).

Note that $f([x, y, z]) < \lambda$, so there are points x', y', z' available to be chosen in Step 3. Moreover, the chosen points are such that $f([x', y, z]) > 0$, and similarly for the others; so, if we do change f, at most one negative value is introduced.

Proposition 6 *If the Markov chain on $\mathcal{P} \cup \mathcal{I}$ is connected, its limiting distribution has a constant value p_0 on elements of $\mathcal{P}$ and a constant value p_1 on elements of $\mathcal{I}$, where $p_1 = (\lambda + 1)^3/(\lambda^3 N)p_0$.*

Proof The first Markov chain is a random walk on a graph in which vertices in $\mathcal{P}$ have degree $N\lambda^3$ (there are N choices of $[x, y, z]$, and λ for each of x', y', z'), while vertices in $\mathcal{I}$ have degree $(\lambda + 1)^3$ (there is only one possible $[x, y, z]$, but $\lambda + 1$ choices for each of x', y', z'). Moreover, the edges of the graph are undirected. (We only have to check this for non-loops; for these the trade of $[x, y, z]$ for $[x', y', z']$ is obviously reversible.) So, as in [11], the result follows from the general theory of Markov chains.

As in the paper of Jacobson and Matthews [11], we define a Markov chain on $\mathcal{P}$ by starting at an element of $\mathcal{P}$ and following the above-defined chain until we return to $\mathcal{P}$. If the earlier Markov chain is connected, then so is this one, and its limiting distribution is uniform on $\mathcal{P}$, by the same arguments as in [11].

Conjecture For any $\mathbf{v}$, $\mathbf{k}$, and λ, the above-defined Markov chain is connected.

Jacobson and Matthews prove this conjecture in the case of Latin squares (with $\mathbf{k} = (1, 1, 1)$ and $\lambda = 1$). Connectedness is known in some other specific cases. The above conjecture is posed for Steiner triple systems, and questions about the rate of convergence are raised, in [5].

The question can be resolved in the case $\mathbf{k} = (1, 1, 1)$ by a simple extension of the argument of Jacobson and Matthews. Here is a sketch of the argument. We can think of one of these designs as a $v \times v$ array with a multiset of symbols of cardinality λ in each cell, so that each symbol occurs once in each row or column. Given two such arrays A and B, we wish to transform A into B by a series of moves. We can suppose that, in the course of these moves, we have changed A so that its first m rows agree with those of B, and we need to confine the effect of all further moves to the last $v - m$ rows.

Suppose that a particular cell, without loss of generality the first cell in row $m + 1$, contains the symbol a more often in A than in B, and the symbol b less often. We want to make an a, b switch. Construct a directed graph whose vertices are the cells, which has an edge between two cells in the same column if the source has more as than bs and the target has more bs than as, and between two cells in the same row if the reverse conditions hold. Clearly all arcs are within the last $v - m$ rows, and a cell is the source of an arc in a column if and only if it is the target of an arc in a row (and *vice versa*. Now this graph must contain a directed cycle whose arcs lie alternately in rows and columns. A sequence of moves switches as and bs on this cycle. We can continue this procedure until an a in the chosen cell is replaced by a b. Continuing in this way we can adjust the $m+1$st row of A to agree with that of B. After $v-1$ such sequences of moves, we have indeed transformed A into B.

We refer to [11] for a more detailed explanation.

The case $\mathbf{k} = (2,1)$ may be intermediate in difficulty between $\mathbf{k} = (1,1,1)$ and $\mathbf{k} = (3)$.

6 A generalisation

In a more general version of the definition, the condition would run as follows. Again we are given a partition $\mathbf{k} = (k_1, \ldots, k_m)$ of k and a positive integer $t < k$. We call an m-tuple $\mathbf{t} = (t_1, \ldots, t_m)$ *admissible* if its sum is t and $0 \le t_i \le k_i$ for $i = 1, \ldots, m$. Now suppose that, for each admissible $\mathbf{t}$, a positive integer $\lambda_{\mathbf{t}}$ is given. Now a t-$(\mathbf{v}, \mathbf{k}, (\lambda_{\mathbf{t}}))$ design of order $\mathbf{v} = (v_1, \ldots, v_m)$ consists of sets $X_1, \ldots, X_m$, with $|X_i| = v_i$ for $i = 1, \ldots, m$,

$$\mathcal{B} \subseteq \binom{X_1}{k_1} \times \cdots \times \binom{X_m}{k_m}$$

with the following property:

> if $\mathbf{t} = (t_1, \ldots, t_m)$ is admissible, then for any choice of sets $T_i \in \binom{X_i}{t_i}$ (for $i = 1, \ldots, m$), there are precisely $\lambda_{\mathbf{t}}$ members $(K_1, \ldots, K_m) \in \mathcal{B}$ for which $T_i \subseteq K_i$ for $i = 1, \ldots, m$.

The additional flexibility makes many more examples possible. We mention a couple of these.

α-resolvable 2-designs Take $\mathbf{k} = (k,1)$, $t = 2$, and let $\lambda_{(2,0)} = \lambda$ and $\lambda_{(1,1)} = \alpha$. Now our design is a 2-(v,k,λ) design whose block set is partitioned into 1-(v,k,α) designs.

Orthogonal arrays over variable-size alphabets Consider the array

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ A & B & C & A & B & C & A & B & C & A & B & C \end{pmatrix}.$$

Any ordered pair of symbols from the appropriate alphabets occurs three times in two of the first three rows, and twice in one of the first three rows and the fourth row. We have a 2-$(\mathbf{v}, (1,1,1,1), (\lambda_{\mathbf{t}}))$ design where $\mathbf{v} = (2,2,2,3)$ and

$$\lambda_{\mathbf{t}} = \begin{cases} 3 & \text{if } \mathbf{t} = (1,1,0,0) \text{ or } (1,0,1,0) \text{ or } (0,1,1,0), \\ 2 & \text{if } \mathbf{t} = (1,0,0,1) \text{ or } (0,1,0,1) \text{ or } (0,0,1,1). \end{cases}$$

A further generalisation would be to consider the analogues of packing (resp. covering) designs replacing the condition that the number of blocks covering the sets $T_1, \ldots, T_m$) is equal to $\lambda_{\mathbf{t}}$ by the condition that it is at most (resp. at least) $\lambda_{\mathbf{t}}$; the natural question is the maximum (resp. minimum) size of such a design.

Further interesting structures are obtained by allowing t to vary. For example, consider resolvable t-designs. If X_1 is the set of points and X_2 a set indexing the resolution classes, then any t points of X_1, or any two points with one in X_1 and one in X_2, are contained in a unique block. I have not attempted to examine this systematically.

References

[1] L. Babai, Almost all Steiner triple systems are asymmetric, *Ann. Discrete Math.* **7** (1980), 37–39.

[2] R. A. Bailey, Latin squares with highly transitive automorphism groups, *J. Austral. Math. Soc.* Series A **33** (1982), 18–22.

[3] D. Bryant and D. Horsley, The embedding problem for partial Steiner triple systems, *Austral. Math. Soc. Gaz.* **33** (2006), 111–115.

[4] P. J. Cameron, *Parallelisms of Complete Designs*, London Math. Soc. Lecture Notes **23**, Cambridge Univ. Press, Cambridge, 1976.

[5] P. J. Cameron (ed.), Problems from the 18th British Combinatorial Conference, *Discrete Math.* **266** (2003), 441–451.

[6] P. J. Cameron, Embedding partial Steiner triple systems so that their automorphisms extend, *J. Combinatorial Designs* **13** (2005), 466–470.

[7] P. C. Clapham, Steiner triple systems with block-transitive automorphism groups, *Discrete Math.* **14** (1976), 121–131.

[8] J. Doyen and R. M. Wilson, Embeddings of Steiner triple systems, *Discrete Math.* **5** (1973), 229–239.

[9] P. B. Gibbons and R. Mathon, The use of hill-climbing to construct orthogonal Steiner triple systems, *J. Combinatorial Designs* **1** (1993), 27–50.

[10] D. G. Higman, Partial geometries, generalized quadrangles and strongly regular graphs, Atti del Convegno di Geometria Combinatoria e sue Applicazioni (Univ. Perugia, Perugia, 1970), pp. 263–293. Ist. Mat., Univ. Perugia, Perugia, 1971.

[11] M. T. Jacobson and P. Matthews, Generating uniformly distributed random Latin squares, *J. Combinatorial Designs* **4** (1996), 405–437.

[12] E. Mendelsohn, On the groups of automorphisms of Steiner triple and quadruple systems, *Congressus Numerantium* **XIII** (1964), pp. 255–264. Utilitas Math., Winnipeg, Man., 1975.

[13] H. J. Ryser, A combinatorial theorem with an application to latin rectangles, *Proc. Amer. Math. Soc.* **2** (1951), 550–552.

[14] A. N. Timashëv, On permanents of random doubly stochastic matrices and on asymptotic estimates for the number of Latin rectangles and Latin squares, *Diskret. Mat.* **14** (2002), 65–86; translation in *Discrete Math. Appl.* **12** (2002), 431–452.

[15] I. M. Wanless, personal communication.

[16] E. J. Williams, Experimental designs balanced for the estimation of residual effects of treatments, *Aust. J. Sci. Res.* Series A **2** (1949), 149–168.

[17] R. M. Wilson, Nonisomorphic Steiner triple systems, *Math. Z.* **135** (1973/74), 303–313.