Semiregular automorphisms of vertex-transitive cubic graphs

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Abstract

An old conjecture of Marušič, Jordan and Klin asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Marušič and Scapellato proved this for cubic graphs. For these graphs, we make a stronger conjecture, to the effect that there is a semiregular automorphism of order tending to infinity with n. We prove that there is one of order greater than 2.

Key words: vertex-transitive graph, semiregular automorphism

1 Introduction

A permutation σ is *semiregular* if all its cycles have the same length. An old conjecture made independently by Marušič, Jordan and Klin (see the introduction to [3] for details) asserts that any finite vertex-transitive graph has a non-trivial semiregular automorphism. Clearly there is no loss of generality in assuming that the graph is connected. Marušič and Scapellato proved:

Theorem 1 *A vertex-transitive connected cubic simple graph has a non-trivial semiregular automorphism.*

The proof allows the possibility that the semiregular element has order 2 or 3. However, in all the examples known to us, there are semiregular elements with order at least 4. We make the following conjecture:

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Conjecture 2 *There is a function f, so that $f(n) \to \infty$ as $n \to \infty$, with the property that a connected vertex-transitive cubic graph on n vertices has a semiregular automorphism of order at least $f(n)$.*

In the rest of this paper, we show that there is always a semiregular automorphism of order at least 3, and end with some examples which give an upper bound to the growth of such a function.

Theorem 3 *Let Γ be a connected vertex-transitive cubic graph. Then Γ has a semiregular automorphism of order greater than 2.*

PROOF. Let Γ be a vertex-transitive cubic graph which has no semiregular automorphism of order greater than 2.

First we show that $\mathrm{Aut}(\Gamma)$ has order divisible by the primes 2 and 3 only. Let σ be an automorphism of prime order greater than 3. If σ fixes a vertex v, then it must fix the three neighbours of v, and then their neighbours, and so on; since Γ is connected, we would find that σ is the identity, a contradiction.

So $|\mathrm{Aut}(\Gamma)| = 2^x 3^y$ for some x, y. We deal with the case that $y = 0$ in the next section. Here we assume that $y > 0$.

If 3 does not divide the order of the vertex stabiliser, then an element of order 3 is semiregular. (Note that in this case we cannot construct a semiregular automorphism of order greater than 3; but such an automorphism will exist unless the exponent of a Sylow 3-subgroup of $\mathrm{Aut}(\Gamma)$ is 3.)

So we may assume that there is an automorphism of order 3 fixing a vertex and permuting its three neighbours transitively. Since Γ is vertex-transitive, it is arc-transitive.

Let v be any vertex, and N a minimal normal subgroup of $G = \mathrm{Aut}(\Gamma)$. We may assume that N is an elementary abelian 2-group. We separate three cases, according to the behaviour of the neighbours of v.

Case 1: The neighbours of v are in the same N-orbit as v. In this case, N is transitive, so Γ is a Cayley graph for N. Since it is a cubic graph, N has three generators, so $|N| \leq 8$; and Γ is either K_4 or the 3-cube. The result is true by inspection.

Case 2: The neighbours of v are all in a single N-orbit which doesn't contain v. In this case, as before, there are just two N-orbits and Γ is bipartite; we find easily that it is the 3-cube.

Case 3: The neighbours of v are all in different N-orbits. In this case, the edges between two orbits (if any) form a 1-factor; the graph $\overline{\Gamma}$ obtained by shrinking each N-orbit to a single vertex and each such 1-factor to a single edge is a cubic vertex-transitive graph, so has a semiregular automorphism of order greater than 2, by induction. This lifts to a semiregular automorphism group of Γ which is not an elementary abelian 2-group, and hence contains an element of order greater than 2. $\square$

2 2-groups

In this section we prove theorem in the case that $\mathrm{Aut}(\Gamma)$ is a 2-group. We begin by analysing three special families of graphs. We state the following three lemmas without proof: the routine arguments required can be supplied by the authors.

Lemma 4 *Let Γ_n be the graph with 2^n vertices shown in Figure 1. If $n \neq 2, 3$ then $\mathrm{Aut}(\Gamma_n)$ is a 2-group with centre of order 2 and exponent 2^{n-1}. In particular, the 2^{n-2}-power of every element of order 2^{n-1} is the central involution of $\mathrm{Aut}(\Gamma_n)$. If H is a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma_n)$ of exponent e then it contains a semiregular element of order e. In particular, $e \geq 2^{n-2}$. Moreover, the stabiliser of a vertex is an elementary abelian 2-group of order at most $2^{2^{n-2}-1}$.*

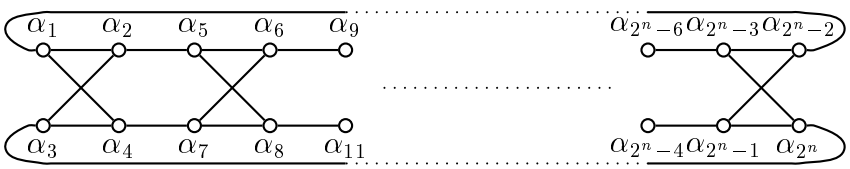


Fig. 1. Vertex-transitive graph Γ_n

Lemma 5 *Let Δ_n be the cubic graph with 2^n vertices shown in Figure 2. The Sylow 2-subgroup S of $\mathrm{Aut}(\Delta_n)$ has exponent 2^{n-1}. If G is a vertex-transitive subgroup of S of exponent e then every element of order e acts semiregularly on Δ_n.*

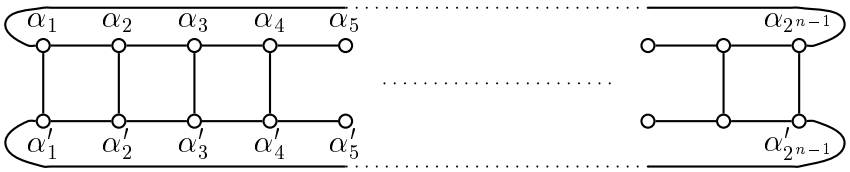


Fig. 2. Vertex-transitive graph Δ_n

Lemma 6 *Let Θ_n be the graph with 2^n vertices shown in Figure 3. The group $\mathrm{Aut}\,\Theta_n$ is a 2-group with exponent 2^{n-1}. If G is a vertex-transitive subgroup of $\mathrm{Aut}\,\Theta_n$ of exponent e then every element of order e acts semiregularly on Θ_n.*

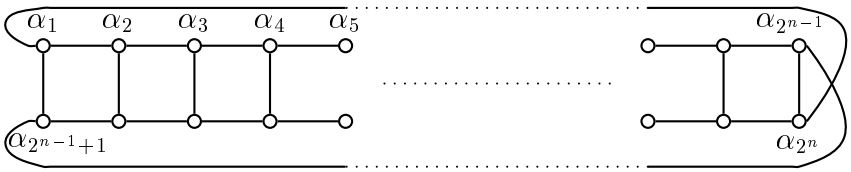


Fig. 3. Vertex-transitive graph Θ_n

Theorem 7 *Let Γ be a connected, cubic, vertex-transitive graph and G be a vertex-transitive 2-subgroup of $\mathrm{Aut}(\Gamma)$. Let e be the exponent of G. Then G contains a semiregular element of order e.*

PROOF. Let $Z = \langle z \rangle$ be a central subgroup of G of order 2 and set

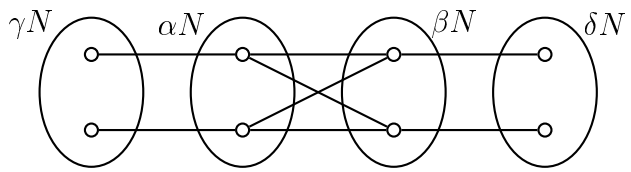
$$N = \bigcap_{g \in G} (ZG_\alpha)^g,$$

where α is a vertex of Γ.

Note that $NG_\alpha = ZG_\alpha$, in particular N is a normal elementary abelian subgroup of G. Furthermore, $\alpha N = \alpha Z$, for every vertex α in Γ.

Case 1: $N_\alpha \neq 1$.

Let $\mathcal{O}$ be the set $\{\alpha N \mid \alpha \in V\Gamma\}$. Every element of $\mathcal{O}$ has size 2. Let α be a vertex of Γ. By connectivity of Γ and normality of N there exists a neighbour β of α such that $\beta N_\alpha = \beta N$. This says that α is joined to every vertex of βN, and vice versa, β is joined to every vertex of αN. So $\alpha N \cup \beta N \cong K_{2,2}$. The graph Γ has valency three, so, α is joined to another vertex, γ. If γ lies in αN then αN and βN contain an edge and so $\Gamma \cong K_4$ and the result is trivial. Therefore we may as well assume that γ does not lie in αN. Furthermore, if $\gamma' \in \gamma N \backslash \{\gamma\}$ then $\{\alpha, \gamma'\}$ is not an edge. A similar argument applies to β. Therefore we get the following picture.



Now, N is a normal subgroup of G. Therefore any result that holds for N_α can be translated to a result for N_γ. This proves that the graph Γ is isomorphic to Γ_n for some n, see Figure 1.

In this case the theorem holds by Lemma 4.

Case 2: $N_\alpha = 1$.

The group N acts semiregularly on Γ. So, by definition of N, we have $N = Z$, hence $G_\alpha Z/Z$ is a core-free subgroup of G/Z.

Let us denote by $\mathcal{O}$ the set of Z-orbits of $V\Gamma$. If for some vertex α we have that two neighbours of α lie in the same Z-orbit then, by an argument similar to the one employed in Case 1 we get that $\Gamma \cong \Gamma_n$ for some n (in this case our theorem holds by Lemma 4). So we may assume that for every vertex α the three neighbours of α lie in three different Z-orbits. On the other hand if a Z-orbit contains an edge of Γ then every Z-orbit contains an edge of Γ. Therefore the graph Γ is isomorphic to Δ_n or to Θ_n for some n. In both cases the theorem holds by Lemma 5 and Lemma 6. It remains to consider the case that $\mathcal{O} = \Gamma/Z$ is a cubic vertex-transitive connected graph and $G/Z \subseteq \mathrm{Aut}(\mathcal{O})$. Now, by induction on n, we have that G/Z contains a semiregular element of order e', where e' is the exponent of G/Z. Now, assume that the exponent e of G equals e'. In this case G/Z contains a semiregular element xZ of order e'. The element x acts semiregularly on Γ/Z, so, x acts semiregularly on Γ and it has order e. It remains to consider the case when $e = 2e'$. Let x be an element of order e in G. We get that $x^{e/2} = x^{e'} = z$ is a central element of G, in particular, x acts semiregularly on Γ. This concludes the proof. $\square$

3 An upper bound

We conclude with an example to show that, if our conjecture is true, the function $f(n)$ cannot grow faster than $n^{1/3}$.

Let p be a prime congruent to ± 1 mod 16, and let G be the group $\mathrm{PSL}(2,p)$. Then G has a maximal subgroup H isomorphic to S_4. This subgroup contains a dihedral subgroup K of order 8, with $N_G(K)$ a dihedral group of order 16. (See Burnside [1] for the subgroups of G.) Then G, acting on the cosets of H, preserves the orbital graph corresponding to the double coset HxH, where $x \in N_G(K) \setminus K$. Since $H \cap x^{-1}Hx = K$, and $|H : K| = 3$, this orbital graph is cubic and 1-transitive. Since $|H| = 3 \cdot 2^3$, it is 4-transitive. Now it follows that G is the full automorphism group. For Tutte's Theorem [4] shows that the full automorphism group has at most twice the order of G. So it contains G as a normal subgroup of index at most 2. If it is larger than G, it would be $\mathrm{PGL}(2,p)$. But this group does not contain a subgroup isomorphic to $S_4 \times C_2$.

Now the largest order of an element of G is p, and the number of vertices of the graph is $(p^3 - p)/48$.

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