The CRC Handbook
of
Combinatorial Designs

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AUTHOR PREPARATION VERSION
25 July 2006
1 Infinite designs

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1.1 Cardinal arithmetic

1.1 Remark In order to motivate the “right” generalisation of $t$-designs to infinite sets, we have to say a bit about the arithmetic of infinite cardinal numbers. We work in Zermelo–Fraenkel set theory with the Axiom of Choice (ZFC). We assume that the definition and basic properties of ordinal numbers are known.

1.2 Definition A cardinal number is an ordinal number which is not bijective with any smaller ordinal number. The cardinality of a set $X$ is the unique cardinal number which is bijective with $X$.

1.3 Example The natural numbers $0, 1, 2, \ldots$ are cardinal numbers; the smallest infinite cardinal number is $\aleph_0$, the cardinality of the set of all natural numbers.

1.4 Theorem Let $\alpha$, $\beta$ be cardinals with $\alpha$ infinite. Then $\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}$. If also $2 \leq \beta \leq 2^\alpha$. Then $\beta^\alpha = 2^\alpha > \alpha$.

1.2 The definition of a $t$-design

1.5 Remark The usual definition of a $t$-design requires some strengthening when extending to infinite sets, if we wish to retain the property that (for example) the complement of a $t$-design is a $t$-design. The authors [?] proposed the following definition, and gave examples to show that relaxing these conditions admits structures which should probably not be called $t$-designs.

1.6 Definition Let $t$ be a positive integer, $v$ an infinite cardinal, $k$ and $\overline{k}$ cardinals with $k + \overline{k} = v$, and $\Lambda$ a $(t+1) \times (t+1)$ matrix with rows and columns indexed by $\{0, \ldots, t\}$ with $(i, j)$ entry a cardinal number if $i + j \leq t$ and blank otherwise. Then a simple infinite $t$-$(v, k, \overline{k}, \Lambda)$ design consists of a set $V$ of points, and a set $B$ of subsets of $V$, having the properties

- $|B| = k$ and $|V \setminus B| = \overline{k}$ for all $B \in B$;
- for $0 \leq i + j \leq t$, let $x_1, \ldots, x_i, y_1, \ldots, x_j$ be distinct points of $V$. Then the number of elements of $B$ containing all of $x_1, \ldots, x_i$ and none of $y_1, \ldots, y_j$ is precisely $\Lambda_{ij}$.
- No block contains another block.

1.7 Remark Sometimes, where there is no confusion, we refer more briefly to a $t$-$(v, k, \lambda)$ design, where $\lambda = \Lambda_{t,0}$, as in the finite case.

1.8 Remark In a non-simple infinite design, repeated blocks are allowed, but the last condition should be replaced by

- No block strictly contains another block.
1.3 Examples and results

1.9 Example A 2-(\(\aleph_0, \aleph_0, \Lambda\)) design: the points are the vertices of the random graph (Rado’s graph), and the blocks are the maximal cliques and co-cliques. Here, \(\lambda_{i,j} = 2^{\aleph_0}\) for all \(i + j \leq 2\), so \(b = r = \lambda > v = k\), where \(b = \lambda_{0,0} = |\mathcal{B}|\), and \(r = \lambda_{1,0}\).

1.10 Definition For \(t\) finite, \(v\) an infinite cardinal, and \(k\) an arbitrary cardinal, an \(S(t, k, v)\) infinite Steiner system consists of a set \(V\) of \(v\) points and a family \(\mathcal{B}\) of \(k\)-subsets of \(V\) called blocks, such that any \(t\) points lie in exactly one block, and (if \(v = k\)) no block contains every point.

1.11 Theorem An infinite Steiner system is an infinite \(t\)-design according to the earlier definition. Moreover, Steiner systems exist for all \(t, k, v\) with \(t < k < v\) and \(v\) infinite. If \(k\) is also finite, then a large set of Steiner systems exists; this is a partition of the set of all \(k\)-subsets into Steiner systems.

1.12 Theorem (Infinite analogue of Fisher’s inequality) Let \(D\) be an infinite \(t\-(v, k, \Lambda)\) design with \(t\) finite. Then \(b = v\), unless \(\lambda\) is infinite and either

- \(\lambda < v\) and \(k > r\) (infinite), in which case, \(b \leq v\); or
- \(\lambda > v\), in which case \(k\) is infinite and \(b > v\).

1.13 Remark There is no “characterization of equality” as in the finite case. Indeed we have the following:

1.14 Theorem Let \(s, t, \lambda, \mu\) be positive integers satisfying

\[t \leq \mu\text{ if and only if } s \leq \lambda.\]

Then there exists a countable design with the properties

(a) every \(t\) points lie in exactly \(\lambda\) blocks;
(b) every \(s\) blocks intersect in exactly \(\mu\) points.

1.15 Example In the case \(s = t = 2, \lambda = \mu = 1\), we have the free projective planes.

1.16 Remark Some remarkable highly symmetric projective planes are constructed using stability theory \([?, ?]\). Space does not permit the discussion of these interesting infinite designs, nor their connections with logic, except for the following consequence.

1.17 Remark There is no infinite analogue of Block’s lemma. In fact, this result does not even hold for infinite Steiner systems:

1.18 Theorem (Evans) \([?]\) Let \(v\) be an infinite cardinal and \(s\) a positive integer. Suppose that \(t \geq 2, s \leq k/t\) and \(k > t\), then there exists an infinite \(S(t, k, v)\) with a block-transitive automorphism group that acts with \(s\) point-orbits.

1.4 See Also

\[?[\] A comprehensive introduction to infinite designs including examples of designs and structures that are not designs; contains much of the information given in this section.
\[?[\] A survey discussing Hrushovski’s amalgamation method and its use in constructing an \(\omega\)-categorical pseudoplane.