Resolutions of the pair design, or 1-factorisations of complete graphs

1 Introduction

A *resolution* of a block design is a partition of the blocks of the design into parallel classes, each of which forms a partition of the points of the design. Usually, studying the resolutions of a design is interlocked with studying the design itself. One case in which this is not so is when the structure of the design is trivial. The most obvious case is where the design is the *pair design*, the 2-$(n,2,1)$ design whose blocks are all the 2-element subsets of the point set. A necessary condition for resolvability is that n is even, since we require partitions of the point set into sets of size 2. We will see that this condition is sufficient and discuss constructions of resolutions, their asymptotic enumeration, resolutions with a high degree of symmetry, and connections with other kinds of structures (notably Latin squares).

Another language is often used here. In a graph G, a 1*-factor* is a set of edges which partitions the vertex set. A 1*-factorisation* is a partition of the edge set into 1-factors. So the objects we are considering are precisely 1-factorisations of the complete graph K_n.

The simplest construction to show that they exist for all even n is geometrical. Let $n = 2k$. Draw a regular $(2k-1)$-gon in the plane, and mark its centre. Now for each edge e of the polygon, consider the following pairs: e and all the diagonals parallel to e; the pair consisting of the vertex opposite e and the centre of the polygon. Each pair occurs precisely once, since given any diagonal, there is a unique edge parallel to it, and given any vertex, there is a unique edge opposite to it.

2 Further constructions

The construction just described can be cast into a more algebraic form. Number the vertices of the polygon by elements $0,1,2,\ldots,2k-2$ of the integers mod $2k-1$ in order, and label the centre as $*$. Now it is easily checked that, for each element a of $\mathbb{Z}/(2k-1)\mathbb{Z}$, there is a 1-factor consisting of all pairs $\{x,y\}$ with $x+y = 2a, x \neq y$ together with the pair $\{a,\infty\}$.

This construction generalises: we can replace $\mathbb{Z}/(2k-1)\mathbb{Z}$ by any abelian group of odd order $n-1$, since in such a group the equation $2x = a$ has a unique solution for any element a.

A second construction from abelian groups uses the elementary abelian group of order $n = 2^d$, the direct sum of d copies of $\mathbb{Z}/2\mathbb{Z}$. This time the 1-factors are indexed by the non-zero group elements, and the 1-factor corresponding to the element c consists of all $\{a,b\}$ for which $a+b=c$. This is sometimes called *affine*, since the 1-factors are the parallel classes of lines in the affine space $\mathrm{AG}(d,2)$ (a line has just two points in this space, so the lines form the 2-$(2^d,2,1)$ design.)

Another construction, which works for numbers n congruent to 2 or 4 mod 6, starts with a Steiner triple system S of order $n-1$ (a 2-$(n-1,3,1)$ design). Take the point set of S together with one additional point ∞; for each element $a \in S$, the corresponding 1-factor consists of the pairs $\{b,c\}$ for which $\{a,b,c\}$ is a block of S, together with $\{\infty,a\}$. If the Steiner triple system consists of the lines of the projective space $\mathrm{PG}(d,2)$, the 1-factorisation is the same as the one above derived from the elementary abelian 2-group. There are many non-isomorphic Steiner triple systems, so this construction gives many different 1-factorisations for appropriate values of n.

An even more prolific construction uses Latin squares. The construction works as follows. First recall that a Latin square of order n is "equivalent" to a 1-factorisation of the complete bipartite graph $K_{n,n}$: if the vertices are $a_1,\ldots,a_n$, $b_1,\ldots,b_n$, then we assign the edge $\{a_i,b_j\}$ to the 1-factor c_k if the (i,j) entry of the Latin square is k. (See the topic essay on *Latin squares: Equivalents and Equivalences* for further details.)

We also note before embarking on the construction that if we have a 1-factorisation of K_{2k} and remove one vertex, what remains is a *near 1-factorisation of K_{2k-1}*: there are $2k-1$ partial 1-factors, each consisting of $k-1$ edges covering all but one vertex; there is a bijection between vertices and partial 1-factors, a vertex corresponding to the partial 1-factor not containing it.

We separate two cases according as n is congruent to 0 or 2 mod 4.

Case 1: $n = 4k$ Take a Latin square of order $2k$, and use it as a 1-factorisation of the complete bipartite graph $K_{2k,2k}$. Now choose two 1-factorisations of K_{2k}, the same or different, with a bijection between the one-factors in each. Place one of these 1-factorisations on each of the parts of the complete bipartite graph, and put together a 1-factor on one side and the corresponding 1-factor on the other side to give a 1-factor on the whole graph.

Case 2: $n = 4k-2$ Once again we take two 1-factorisations of K_{2k}. Remove a point from each, and choose a bijection between the sets of partial 1-factors in the remaining structures; put together the corresponding partial

1-factors. Each of the resulting partial 1-factors will omit two points, viz., corresponding points on the two sides; add this pair of points to produce a complete 1-factor.

Now take a Latin square of order $2k-1$, and use it as a 1-factorisation of $K_{2k-1,2k-1}$. Select a 1-factor; remove it, and match up the two bipartite blocks with the point sets of the K_{2k-1}s so that corresponding points would have been joined by edges of the missing 1-factor. The remaining $2k-2$ 1-factors complete the 1-factorisation of K_{4k-2}.

This construction shows that the number of 1-factorisations of K_n for even n is very roughly the square root of the number of Latin squares of order n. More precisely, the logarithms of the numbers of 1-factorisations and Latin squares are $\frac{1}{2}n^2\log n$ and $n^2\log n$ respectively.

Further details of this material can be found in [1].

3 1-factorisations and Latin squares

We have seen that Latin squares can be used to construct many 1-factorisations. Conversely, any 1-factorisation gives a special kind of Latin square, namely, a symmetric Latin square with constant diagonal, as follows.

Number the 1-factors as $c_1,\ldots,c_{n-1}$. Now construct a Latin square as follows: the (i,j) entry is equal to n if $i=j$, and to k if $i\neq j$ and $\{i,j\}$ lies in the 1-factor c_k.

Conversely any symmetric Latin square with constant diagonal (which we can take to have entries n, by permuting the entries if necessary) arises from a 1-factorisation by this construction.

Another way of regarding this is as follows. Each 1-factor can be thought of as a fixed-point-free involution (a permutation all of whose cycles have length 2): the cycles of the permutation are the edges of the 1-factor. Now take these permutations together with the identity, and write them in "list form" (that is, permutation σ is represented as $(a_1,\ldots,a_n)$ if σ maps i to a_i for $i=1,\ldots,n$. Now these lists form the rows of a Latin square. This is a conjugate of the one constructed above, since the square just defined has (i,j) entry k if $\{j,k\}$ is an edge in c_i for $i<n$, and (n,j) entry j.

4 Small cases

4.1 The case $n = 4$

The 1-factorisation in this case is absolutely unique, since there are only three ways of partitioning $\{1,2,3,4\}$ into two pairs, namely $\{\{1,2\},\{3,4\}\}$, $\{\{1,3\},\{2,4\}\}$ and $\{\{1,4\},\{2,3\}\}$. This structure is the affine plane of order 2 with its usual parallelism, or can be regarded as derived from the $Z/3\mathbb{Z}$ or the trivial Steiner triple system of order 3 by our earlier constructions.

4.2 The case $n = 6$

On a set of six points, there are 15 possible 1-factors: the three pairs can be chosen in $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$ ways, but we must divide by the $3! = 6$ possible orders in which the pairs could have been chosen.

Sylvester discovered in the nineteenth century that there are precisely six 1-factorisations, all isomorphic. We outline the argument.

The first two 1-factors together form a hexagon. Any additional 1-factor must use either the three long diagonals of the hexagon, or one long and two short diagonals. Since there are altogether three long and six short diagonals, the second case must occur always, and the remaining three 1-factors are uniquely determined. Hence two disjoint 1-factors can be completed to a 1-factorisation in a unique way. Now the number of choices of two disjoint 1-factors is $15 \cdot 8 = 120$, while there are $5 \cdot 4 = 20$ ways of saying which are the first two 1-factors in a given 1-factorisation; so there are $120/20 = 6$ different 1-factorisations, all isomorphic.

Sylvester called edges, 1-factors and 1-factorisations *duads*, *synthemes*, and *synthematic totals* respectively.

This remarkable object is connected with the exceptional outer automorphism of the symmetric group of degree 6, and has been used to construct many other combinatorial structures including the projective plane of order 4, the 5-$(12,6,1)$ Steiner system, and the Moore graph of valency 7 on 50 vertices. See [3] for details.

5 Highly symmetric 1-factorisations

An *automorphism* (sometimes called a *weak automorphism*) of a 1-factorisation is a permutation of the points which carries the set of 1-factors into itself. If every

1-factor is fixed, then we have a *strong automorphism*.

Not surprisingly, the weak automorphisms form a group G (a subgroup of the symmetric group S_n), while the strong automorphisms form a normal subgroup N of G. The group N is an elementary abelian 2-group which acts semiregularly, but the structure of G may be more interesting.

In the case of the affine 1-factorisations, the group of weak automorphisms is the affine group $\mathrm{AGL}(d,2)$, while the strong automorphisms are precisely the translations of the affine space.

For $n=4$, the uniqueness of the 1-factorisation shows that any permutation is an automorphism, so the (weak) automorphism group is the symmetric group S_4. This group has a normal subgroup N of order 4, the *Klein group*, consisting of the identity and the three permutations corresponding to the three 1-factors; this is the strong automorphism group. The automorphism group permutes the three 1-factors arbitrarily, in accordance with the isomorpism $S_4/N \cong S_3$. Note that $\mathrm{AGL}(2,2) \cong S_4$.

For $n=6$, Sylvester's result shows that the (weak) automorphism group is a subgroup of index 6 in the symmetric group which acts on the five 1-factors as the symmetric group S_5. Its action on the points is that of the *linear fractional group* $\mathrm{PGL}(2,5)$, which happens to be isomorphic to S_5, and is triply transitive on the six points. The group of strong automorphisms is trivial in this case.

Using the classification of the doubly transitive permutation groups (a consequence of the classification of finite simple groups), Cameron and Korchmàros [2] showed:

Theorem 1 *A* 1*-factorisation of* K_n *which admits a doubly transitive group of weak automorphisms must be one of the following: the affine* 1*-factorisation with* $n=2^d$*; or a unique example for each of the values* $n=6$*,* 12*,* 28.

In the other direction, Wanless *et al.* have recently published a proof of the folklore theorem that almost all 1-factorisations have trivial (weak) automorphism groups.

References

[1] P. J. Cameron, *Parallelisms of Complete Designs*, London Math. Soc. Lecture Notes **23**, Cambridge University Press 1976.

[2] P. J. Cameron and G. Korchmàros, One-factorizations of complete graphs with a doubly transitive automorphism group, *Bull. London Math. Soc.* **25** (1993), 1–6.

[3] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Math. Soc. Student Texts **22**, Cambridge University Press 1991.

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