

Association schemes

1 The definition

The origins of association schemes lie in statistics, in the work of R. C. Bose and his co-workers. The subject has close connections with permutation groups (D. G. Higman, B. Weisfeiler), coding theory (P. Delsarte), and even knot theory (F. Jaeger).

Combinatorially, an association scheme is a set $\{C_0, C_1, \dots, C_r\}$ of binary relations on a set Ω (that is, subsets of the set Ω^2 of ordered pairs of elements of Ω), satisfying the following four conditions:

(AS1) $\{C_0, C_1, \dots, C_r\}$ is a partition of Ω^2 .

(AS2) C_0 is the diagonal $\{(x, x) : x \in \Omega\}$ of Ω^2 .

(AS3) For $i = 0, \dots, r$, the relation C_i is symmetric, that is, $(x, y) \in C_i$ implies $(y, x) \in C_i$.

(AS4) Given $i, j, k \in \{0, \dots, r\}$ and $(x, y) \in C_k$, the number of points $z \in \Omega$ such that $(x, z) \in C_i$ and $(z, y) \in C_j$ depends only on i, j, k , not on x, y . (We write this number as p_{ij}^k .)

The relations C_0, \dots, C_r are the *associate classes* of the scheme; two points x and y are *i th associates* if $(x, y) \in C_i$.

A binary relation C on Ω can be represented by a zero-one matrix whose rows and columns are indexed by Ω , having (x, y) entry 1 if $(x, y) \in C$, and 0 otherwise. If we translate the relations of an association scheme into matrices in this way, we obtain zero-one matrices A_0, \dots, A_r satisfying the following conditions:

(AS1) $A_0 + \dots + A_r = J$, where J is the all-1 matrix.

(AS2) $A_0 = I$, where I is the identity matrix.

(AS3) For all i , $A_i^\top = A_i$.

(AS4) There are numbers p_{ij}^k such that

$$A_i A_j = \sum_{k=0}^r p_{ij}^k A_k,$$

in other words, the span of A_0, \dots, A_r is an algebra.

This is often taken as the definition of an association scheme. The algebra referred to in (AS4) is the *Bose–Mesner algebra* of the scheme.

There are many good references to association schemes; books on the topic include Bannai and Ito [3], Brauer, Cohen and Neumaier [5], Godsil [8], and Bailey [2].

2 Examples

There are many examples of association schemes. For example, a graph is called *strongly regular* if the relations of equality, adjacency and non-adjacency form an association scheme. Many such graphs exist, and there are extensive catalogues.

The *Johnson scheme* $J(v, k)$ has as points the k -element subsets of a v -element set, where $k \leq \frac{1}{2}v$; two points are i th associates if they intersect in $k - i$ points.

The *Hamming scheme* $H(n, q)$ has as points all n -tuples over an alphabet of cardinality q ; two points are i th associates if they agree in $n - i$ positions.

Delsarte [7] gave a detailed treatment of subsets of association schemes; his results for Hamming and Johnson schemes have important applications in coding theory and design theory respectively.

The *cyclotomic scheme* $C(q, r)$, where r divides $q - 1$ (and the quotient is even if q is odd) has as points the elements of the finite field $\text{GF}(q)$. Numbering the cosets of the subgroup of order $(q - 1)/r$ in the multiplicative group of the field as K_1, \dots, K_r , with $K_0 = \{0\}$, the points x and y are i th associates if $x - y \in K_i$.

Many other examples are derived from permutation groups, as described below.

One final example is the *rectangular* association scheme, whose points are the cells of an $m \times n$ grid; the associate classes are “equal”, “same row”, “same column”, and “other”.

3 Permutation groups

A permutation group G acting on a set Ω is said to be *generously transitive* if, given any two elements x and y of Ω , there is a permutation g in G which interchanges x and y . In particular, we see that a generously transitive group is transitive.

There is a natural action of G on the set Ω^2 ; so Ω^2 is partitioned into orbits of G . The generous transitivity now guarantees that the orbit partition satisfies (AS2)

and (AS3). The fact that the parts are orbits guarantees that (AS4) also holds: for, if $(x, y), (x', y') \in C_k$, where C_k is a part of the partition, then there exists $g \in G$ mapping (x, y) to (x', y') ; then g maps the set $\{z : (x, z) \in C_i, (z, y) \in C_j\}$ to the corresponding set involving x' and y' , so the numbers of such points are equal.

Thus, every generously transitive group gives rise in a natural way to an association scheme. There are very many examples of such groups!

Weaker conditions than generous transitivity suffice for a permutation group to act on an association scheme; it is necessary to take unions of orbits as associate classes. This is explored in [1]. More background on permutation groups can be found in [6].

4 The Bose–Mesner algebra

If the product of two real symmetric matrices is symmetric, the matrices commute. Hence the Bose–Mesner algebra is commutative. The spectral theorem for real symmetric matrices implies that the basis matrices can be simultaneously diagonalised by an orthogonal transformation.. That is, there is an expression

$$\mathbb{R}^\Omega = V_0 \oplus \cdots \oplus V_r$$

of \mathbb{R}^Ω as a direct sum of orthogonal subspaces such that each basis matrix is a scalar on each V_i . The subspaces V_i are the *strata*.

Let E_i be the orthogonal projection onto the subspace V_i . Then the matrices E_0, \dots, E_r form an alternative basis for the Bose–Mesner algebra, consisting of pairwise orthogonal idempotents. If $A_i = \sum P_i(j)E_j$, then the numbers $P_i(j)$, for $j = 0, \dots, r$, are the eigenvalues of A_i . (Thus P_i is a character of the algebra.)

The dimensions of the subspaces V_i (that is, the ranks of the E_i) are algebraic functions of the combinatorial parameters p_{ij}^k . The fact that they are positive integers gives necessary conditions (the *integrality conditions*) for the existence of an association scheme with specified parameters.

Further necessary conditions can be derived from the fact that the Bose–Mesner algebra is closed under pointwise (Hadamard) product, since it has a basis of zero-one matrices. The Hadamard product $E_i \circ E_j$ of idempotents is positive semi-definite, so its eigenvalues (which can be computed) are non-negative.

5 Partial balance

An incomplete-block design is said to be *partially balanced* with respect to an association scheme on the set of points if the number of blocks incident with two points x and y depends only on the associate class containing the pair (x, y) . Such a design is called a PBIBD for short.

Partially balanced designs were introduced by Bose and Nair [4]. They have many of the advantages of BIBDs but, unlike the latter, are not bound by Fisher's Inequality asserting that there must be at least as many blocks as treatments. We cannot describe here in detail the uses of partially balanced incomplete-block designs in statistics – see Bailey [2] for this. We mention two types of examples of such designs.

A 2-design (balanced incomplete-block design) is said to be *quasi-symmetric* if the number of points incident with two distinct blocks takes just two different values. See [11] for a survey of such designs. Now the dual of a quasi-symmetric BIBD is a PBIBD with respect to an association scheme on the set of blocks, where the associate classes are defined by the sizes of the intersections of the blocks.

Let G be a generously transitive permutation group on Ω , and H an intransitive subgroup of G . Taking blocks to be the translates of any union of orbits of H , we obtain a design which is partially balanced with respect to the association scheme defined by the orbits of G .

6 Coherent configurations

If G is an arbitrary permutation group on Ω (not necessarily generously transitive), then the orbits of G on Ω^2 form a partition, which satisfies conditions (AS1) and (AS4), but only weaker versions of the other two conditions. To model this situation, a *coherent configuration* is defined to be a set C_0, \dots, C_r of binary relations on Ω satisfying the following conditions:

(CC1) $\{C_0, C_1, \dots, C_r\}$ is a partition of Ω^2 .

(CC2) The diagonal $\{(x, x) : x \in \Omega\}$ of Ω^2 is the union of a subset of $\{C_0, \dots, C_r\}$.

(CC3) For $i = 0, \dots, r$, there exists j such that $\{(y, x) : (x, y) \in C_i\} = C_j$.

(CC4) Given $i, j, k \in \{0, \dots, r\}$ and $(x, y) \in C_k$, the number of points $z \in \Omega$ such that $(x, z) \in C_i$ and $(z, y) \in C_j$ depends only on i, j, k , not on x, y . (This is identical with (AS4).)

Thus, the orbits of any permutation group form a coherent configuration. The last condition again implies that the matrices of the relations span an algebra, called the *basis algebra* of the configuration. Unlike for association schemes, this algebra is not in general commutative, so its representation theory is more complicated.

A coherent configuration is *homogeneous* if, in place of (CC2), it satisfies the stronger condition (AS2). Thus, the orbits of a transitive permutation group form a homogeneous coherent configuration. Hanaki and Miyamoto [9] have determined all homogeneous coherent configurations on up to 30 points.

A completely different generalisation of association schemes has been proposed in statistics, where symmetry of the matrices is more important than closure under multiplication. The set of real symmetric matrices is closed under the operation of *Jordan product* $A * B = \frac{1}{2}(AB + BA)$. Thus, a *Jordan scheme* is a partition of Ω^2 satisfying (AS1)–(AS3), but with (AS4) (closure under multiplication) replaced by closure under Jordan product. Then the span of the basis matrices is a Jordan algebra, and we must use the structure theory for these, rather than associative algebras. See [10].

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