Modular Lie powers of relation modules and free central extensions of groups

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Let G be a group given by a free presentation $G = F/N$ and consider the quotient

$$F/[\gamma_c N, F] \tag{1}$$

where $\gamma_c N$ denotes the c-th term of the lower central series of N and $c \geq 2$. This quotient is a free central extension of $F/\gamma_c N$, which is in its turn an extension of $G = F/N$ with free nilpotent kernel $N/\gamma_c N$. While the quotient $F/\gamma_c N$ is always torsion-free, elements of finite order may occur in $\gamma_c N/[\gamma_c N, F]$, the centre of (1). In the case where $c = 2$ and $R = F'$, that is for the free centre-by-metabelian group $F/[F'', F]$, this was discovered by C.K. Gupta who proved in 1973 that the free centre-by-metabelian group of rank $n \geq 4$ contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre. It turned out that for certain values of c the torsion subgroup $\tau_c N$ of (1), which is of course contained in the central subgroup $\gamma_c N/[\gamma_c N, F]$, can be identified with a homology group: *Suppose that G has no elements of order dividing c, then*

$$\tau_c N \cong \begin{cases} H_4(G, \mathbb{Z}_c), & \text{if } c \text{ is a prime;} \\ H_6(G, \mathbb{Z}_2), & \text{if } c = 4. \end{cases} \tag{2}$$

These results were established in the eighties and nineties. More recently, new results by Bryant, Erdmann and Schocker on modular Lie representations made it possible to make further progress on the torsion subgroup of (1). In joint work with Marianne Johnson we have been able to identify this torsion subgroup for $c = 6$ (for G without elements of order 2 and 3), and very recently we extended this to the case where c is an arbitrary natural number that is not a prime power. The result was a big surprise to us and it is in startling contrast to the results (2). In my talk I will review the history of the problem, and then I will focus on these recent developments.