A new family of modules with 2-dimensional 1-cohomology

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Abstract

It is known that for finite simple groups it is possible for a faithful absolutely irreducible module to have 1-cohomology of dimension at least 3. However, even faithful absolutely irreducible modules with 2-dimensional 1-cohomology are rare. We exhibit a new infinite family of such modules.

1 Introduction

Over the past few decades, a number of first cohomology groups have been computed, see for example [1, 2, 8, 5, 6]. The first cohomology group $\mathrm{H}^1(G;M)$ of a group G at a module M has two fundamental interpretations: first, it parametrises the complements of M in the split extension $M{:}G$ of M by G; and second, it parametrises the extensions of M by the trivial G-module. In what follows we assume that M is a faithful absolutely irreducible module for G, since $\dim \mathrm{H}^1(G;M)$ can be made arbitrarily large if either of these conditions fails.

In most cases which have been calculated up to now, $\mathrm{H}^1(G;M)$ has dimension 0 or 1, and in the few remaining cases it has dimension 2 or 3. The known examples of 3-dimensional 1-cohomology are given in Bray and Wilson [4] and Scott [9]. The two examples of [4] are explicit, whilst despite the infinitude of examples in [9], the results therein do not allow one to give an explicit example of a module with 3-dimensional 1-cohomology.

The results of this paper were discovered when I was carrying out a systematic calculation of 1-cohomologies (and 2-cohomologies) of (absolutely) irreducible modules of simple (and almost simple) groups, in connection with the Web-Atlas project [10]. Many of the 2-dimensional 1-cohomologies seen have an exceptional feel to them, such as $\dim \mathrm{H}^1(\mathrm{A}_6; \mathbb{F}_3^{\,4}) = 2$ (the smallest such example). During the course of this search I found that $\dim \mathrm{H}^1(\mathrm{L}_3(7); \mathbb{F}_3^{\,55}) = 2$, while for other irreducible $\mathrm{L}_3(7)$-modules the 1-cohomology is at most 1-dimensional. This looked odd, especially when you compare the complexity of the Sylow 3-subgroup (3^2) with that of the Sylow 7-subgroup (7_+^{1+2}), and led to the Main Theorem. The smallest example which is part of this series, namely that $\dim \mathrm{H}^1(\mathrm{L}_3(4); \mathbb{F}_3^{\,19}) = 2$, was discovered earlier but somehow did not raise any eyebrows.

Main Theorem. Let $n \geqslant 3$, let p be a prime such that $p \mid n$, let q be a prime power such that $q \equiv 1 \pmod p$, and let k be a field of characteristic p. Then $\dim_k \mathrm{H}^1(\mathrm{L}_n(q); M) = 2$ where M is a certain absolutely irreducible $k\mathrm{L}_n(q)$-module of dimension $\frac{q^n-1}{q-1} - 2$. We obtain M as the non-trivial composition factor of the permutation module of degree $\frac{q^n-1}{q-1}$ for $\mathrm{L}_n(q)$ over k, corresponding to the action of $\mathrm{SL}_n(q)$ on the projective points (or hyperplanes) of the natural n-dimensional $\mathbb{F}_q\mathrm{SL}_n(q)$-module.

In addition to the above family of cross characteristic examples, the following examples of 2-dimensional 1-cohomology in defining characteristic are also known. Cline, Jones, Parshall and Scott [8, 5] give the examples of $\Omega_{4m}^{\varepsilon}(q)$ for q even, $q > 2$ and $m \geqslant 2$ acting on a module M of dimension $2m(4m-1)-2$, where M is the non-trivial composition factor of $\Lambda^2(V)$, with V being a natural module of $\Omega_{4m}^{\varepsilon}(q)$. They also point out that the 'corresponding' (i.e. 26-dimensional) module of ${}^3\mathrm{D}_4(q)$ for q even, $q > 2$ also has 2-dimensional 1-cohomology. The above results probably hold when $q = 2$. For the groups $\Omega_{4m}^{+}(2)$, $m \geqslant 2$ see [8, 5], while we have done explicit computations for the relevant modules of $\Omega_{4m}^{-}(2)$ for $4m \in \{8, 12, 16, 20, 24, 28\}$ and ${}^3\mathrm{D}_4(2)$. In all cases, a subquotient of the tensor square of the natural module exhibits the 2-dimensional 1-cohomology.

Other than the above families of examples we know explicitly of only finitely many faithful absolutely irreducible modules for which the 1-cohomology is at least 2-dimensional.

2 Proof of the Main Theorem

Firstly, we establish some notation. The Atlas [7] notation is used for group structures, with E_q denoting an elementary abelian group of order q. For G a group and k a field, we also use k to denote the trivial kG-module. If V, U and W are modules, we write $V \cong U.W$ to mean that V has a submodule (isomorphic to) U and $V/U \cong W$; and we write $V \cong U \cdot W$ if $V \cong U.W$ and U does not have a complement isomorphic to W. The standard results we quote here can be found, for example, in Benson's book [3] (which states them in more generality).

Our example involves a certain module of $\mathrm{L}_n(q)$ in characteristic p, where $n \geqslant 3$, $p \mid n$ and $q \equiv 1 \pmod p$. We first recall some well-known facts about the groups $\mathrm{L}_n(q) = \mathrm{PSL}_n(q)$

and $\mathrm{SL}_n(q)$. Firstly, for $n \geqslant 2$ and $(n, q) \neq (2, 2)$ or $(2, 3)$ the groups $\mathrm{L}_n(q)$ are simple and the groups $\mathrm{SL}_n(q)$ are perfect and quasi-simple. Secondly, the Schur multiplier of $\mathrm{SL}_n(q)$ is always trivial, except for the cases $(n, q) = (2, 4)$, $(2, 9)$, $(3, 2)$, $(3, 4)$ or $(4, 2)$, and the r'-part of the Schur multiplier of $\mathrm{SL}_n(q)$ is always trivial where r is the prime such that $r \mid q$.

It is known that the stabiliser of a projective point (i.e. 1-space) in $\mathrm{SL}_n(q)$ is a subgroup H of shape ${\mathrm{E}_q}^{n-1}{:}\mathrm{GL}_{n-1}(q)$, which contains a subgroup ${\mathrm{E}_q}^{n-1}{:}\mathrm{SL}_{n-1}(q) \cong \mathrm{ASL}_{n-1}(q)$ to index $q-1$. We note that H need not be isomorphic to $\mathrm{AGL}_{n-1}(q)$; in particular, we have non-isomorphism in the cases of interest.[1] Since n and $n-1$ are coprime, the subgroup of scalars of $\mathrm{SL}_n(q)$ intersects the said subgroup $\mathrm{ASL}_{n-1}(q)$ trivially, and thus the image of this subgroup in any of the images $C.\mathrm{L}_n(q)$ of $\mathrm{SL}_n(q)$ is still isomorphic to $\mathrm{ASL}_{n-1}(q)$. Thus any non-trivial representation of $\mathrm{SL}_n(q)$ restricts faithfully to the subgroup $\mathrm{ASL}_{n-1}(q)$.

So now let $n \geqslant 3$, $p \mid n$, $p \mid (q-1)$, $G \cong \mathrm{SL}_n(q)$, $H \cong {\mathrm{E}_q}^{n-1}{:}\mathrm{GL}_{n-1}(q)$ (the point stabiliser), and let k be a field of characteristic p. Let P be the permutation module over k of G on the cosets of H. Thus P is obtained by inducing the trivial kH-module up to G. Since $\mathrm{Z}(G) \leqslant H$, $\mathrm{Z}(G)$ acts trivially on P, and thus P is a (permutation) module for $\mathrm{L}_n(q)$. The dimension of P is $m := \frac{q^n-1}{q-1}$. Permutation modules for transitive groups have a unique trivial submodule, U say, generated by the all 1s vector $u = \sum_{i=1}^m e_i$; and a unique trivial quotient, whose kernel is the augmentation submodule, W say, where $W = \{\sum_{i=1}^m a_i e_i : a_i \in k \mid \sum_{i=1}^m a_i = 0\}$. We have $U \leqslant W$ since $p \mid m$, and we let $M := W/U$. Now $\mathrm{SL}_n(q)$ is perfect and so there are no non-split modules of type $k \cdot k$ for $\mathrm{SL}_n(q)$, and thus M has no trivial submodules or quotients. From Clifford Theory we find that a faithful representation of $\mathrm{ASL}_{n-1}(q)$ in characteristic p has dimension at least $q^{n-1}-1$, since $\mathrm{SL}_{n-1}(q)$ has a single orbit on non-zero vectors of (the dual of) its natural module. But $2(q^{n-1}-1) > \dim M = \frac{q^n-1}{q-1} - 2$ and so M is absolutely irreducible. Therefore P is a uniserial module of shape $k \cdot M \cdot k$.

Now $\mathrm{SL}_n(q)$ is perfect, and so $\dim \mathrm{H}^1(\mathrm{SL}_n(q); k) = 0$, and $\dim \mathrm{H}^2(\mathrm{SL}_n(q); k)$ is the p-rank of the Schur multiplier of $\mathrm{SL}_n(q)$, which is 0. Now $\dim \mathrm{H}^1(H; k) = 1$, which is the p-rank of H/H'. The Eckmann–Shapiro Lemma then implies that $\dim \mathrm{H}^1(\mathrm{SL}_n(q); P) = 1$. A well-known long exact sequence of cohomologies, applied to the permutation module $P \cong k \cdot M \cdot k$, with (trivial) submodule $U \cong k$ and quotient $Q \cong M \cdot k$ is given below. The dimensions of these cohomology groups for $\mathrm{SL}_n(q)$ are given below. (The dimensions for $\mathrm{L}_n(q)$ differ from these, and also depend on whether the p-part of $q-1$ is greater than the p-part of n.)

$$\begin{array}{ccccccccccccccccc} 0 & \to & \mathrm{H}^0(k) & \to & \mathrm{H}^0(P) & \to & \mathrm{H}^0(Q) & \to & \mathrm{H}^1(k) & \to & \mathrm{H}^1(P) & \to & \mathrm{H}^1(Q) & \to & \mathrm{H}^2(k) & \to & \cdots \\ 0 & & 1 & & 1 & & 0 & & \mathbf{0} & & \mathbf{1} & & a(=1) & & \mathbf{0} & & \end{array}$$

[1]We note that the action of $\mathrm{GL}_{n-1}(q)$ on ${\mathrm{E}_q}^{n-1}$ is the [dual of] the tensor product of the natural representation and the determinant representation. As a result $H \cong {\mathrm{E}_q}^{n-1}{:}\mathrm{GL}_{n-1}(q)$ is isomorphic to $\mathrm{AGL}_{n-1}(q)$ if and only if H has trivial centre, which is if and only if $(n, q-1) = 1$. In the cases of interest, we have $(n, q-1) \neq 1$, and thus $H \ncong \mathrm{AGL}_{n-1}(q)$.

We are interested in $a := \dim \mathrm{H}^1(\mathrm{SL}_n(q); Q)$, and the emboldened figures, which we justified earlier, ensure that $a = 1$. There is no module $k \cdot k$ for $\mathrm{SL}_n(q)$ and thus $\dim \mathrm{H}^1(\mathrm{SL}_n(q); M) = 2$. But any module $M \cdot k$ for $\mathrm{SL}_n(q)$ actually represents the quotient group $\mathrm{L}_n(q)$, since k is not a submodule of $M \otimes k^* \cong M$. (Alternatively, note that the centraliser algebra of the module $M \cdot k$ consists just of scalar matrices. Thus central elements of $\mathrm{SL}_n(q)$ act as scalars on $M \cdot k$, and since they act trivially on the quotient k of this, they act trivially on $M \cdot k$.) Therefore we conclude that $\dim \mathrm{H}^1(\mathrm{L}_n(q); M) = 2$, thus completing the proof of the Main Theorem.

The corresponding construction does not work when $n = 2$ (and thus $p = 2$). This is because the module M is not absolutely irreducible, splitting into two (absolutely irreducible) non-isomorphic summands of dimension $\frac{1}{2}(q-1)$. This splitting always occurs over $\mathbb{F}_4$, and will even occur over $\mathbb{F}_2$ if $q \equiv \pm 1 \pmod 8$. The above argument gives $\dim \mathrm{H}^1(\mathrm{L}_2(q); M) = 2$, and thus each constituent of M has 1-dimensional 1-cohomology.

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