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ON THE ORDERS OF AUTOMORPHISM GROUPS OF FINITE GROUPS. II

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Abstract

In the Kourovka Notebook, Deaconescu asks if $|\mathrm{Aut}\, G| \geqslant \phi(|G|)$ for all finite groups G, where ϕ denotes the Euler totient function; and whether G is cyclic whenever $|\mathrm{Aut}\, G| = \phi(|G|)$. In an earlier paper we have answered both questions in the negative, and shown that $|\mathrm{Aut}\, G|/\phi(|G|)$ can be made arbitrarily small. Here we show that these results remain true if G is restricted to being perfect, or soluble.

1. *The question, and general overview*

Let ϕ denote the Euler totient function, so that $\phi(n)$ is the number of integers m with $1 \leqslant m \leqslant n$ such that m and n are coprime, and

$$\frac{\phi(n)}{n} = \prod_{i=1}^{r} \frac{p_i - 1}{p_i},$$

where $p_1 < p_2 < \ldots < p_r$ are the prime factors of n. It is easy to see that for finite abelian groups G, we have $|\mathrm{Aut}\, G| \geqslant \phi(|G|)$, with equality if and only if G is cyclic.

In [**1**] we showed that neither statement holds for arbitrary finite groups, thus solving Problem 15.43 of the Kourovka Notebook [**5**].

On the other hand they hold (trivially) for finite simple groups (indeed for all finite groups with trivial centre), and one is led to ask: For what classes of finite groups do the statements hold?

A long-standing conjecture of Schenkman [**6**], that if G is a finite non-cyclic p-group of order at least p^3 then $|G|$ divides $|\mathrm{Aut}\, G|$, would imply that both statements hold for finite nilpotent groups. Indeed, this is known to hold for nilpotent groups of class 2, see Schenkman [**6**].

In this paper we show that the statements do not hold for the class of perfect groups, nor for the class of soluble groups. As in [**1**], we actually prove stronger results:

Theorem 1. *For all $\varepsilon > 0$ there exists a finite perfect group G such that $|\mathrm{Aut}\, G| < \varepsilon.\phi(|G|)$.*

Theorem 2. *For all $\varepsilon > 0$ there exists a finite soluble group G such that $|\mathrm{Aut}\, G| < \varepsilon.\phi(|G|)$.*

Theorem 3. *For all $N \in \mathbb{N}$ there exists a finite perfect group G with $|G| > N$ such that $|\mathrm{Aut}\, G| = \phi(|G|)$.*

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Theorem 4. *For all $N \in \mathbb{N}$ there exists a finite non-cyclic soluble group G with $|G| > N$ such that $|\mathrm{Aut}\, G| = \phi(|G|)$.*

We were unable to resolve the case of supersoluble groups, but are marginally inclined to the view that:

Conjecture. *If G is a finite non-nilpotent supersoluble group, then $|\mathrm{Aut}\, G| > \phi(|G|)$.*

Conventions. Throughout this paper, we shall only consider finite groups. The notation for group structures is based on that used in the Atlas [**3**]. The notation $\mathrm{O}_p(G)$, $\mathrm{O}_{p'}(G)$, $\mathrm{O}^p(G)$, $\mathrm{Aut}\, G$, and $\mathrm{Out}(G)$ is standard. The abbreviation PIM stands for projective indecomposable module. If U and V are modules then $U \cdot V$ denotes a non-split extension of U by V with U being the submodule and V being the quotient.

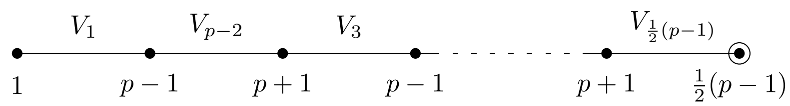
2. *Some modules and cohomology*

We need some information about modules and cohomology of $\mathrm{L}_2(p)$, especially when $p \equiv 7 \pmod 8$. The following information was established in [**1**]:

Lemma 5. *For p prime and $p \equiv 7 \pmod 8$ there are precisely two isomorphism classes of $\mathbb{F}_2\mathrm{L}_2(p)$-modules $1 \cdot U$ in which U is absolutely irreducible of dimension $\frac{1}{2}(p-1)$, and the 1 denotes the trivial module. These two modules are interchanged by the non-trivial outer automorphism of $\mathrm{L}_2(p)$, and both of these modules have zero 1-cohomology. These two modules have the forms $1 \cdot U_1$ and $1 \cdot U_2$ where U_1 and U_2 are not isomorphic.*

For all primes p there are just p irreducible modules of $\mathrm{SL}_2(p)$ in characteristic p. Their dimensions are all different, and at most p, and we label the $\mathrm{SL}_2(p)$-irreducible of dimension i ($1 \leqslant i \leqslant p$) as V_i. For p odd, the central involution of $\mathrm{SL}_2(p)$ acts trivially on V_i if and only if i is odd; in such cases we regard V_i as being an $\mathrm{L}_2(p)$ module. Of course, V_1 is the trivial module for $\mathrm{L}_2(p)$.

For $p \equiv 3 \pmod 4$ the Brauer tree of the principal block of $\mathrm{L}_2(p)$ in characteristic p is a straight line with $\frac{1}{2}(p+1)$ nodes and diagram



where we have labelled the nodes with the degrees of ordinary characters to which they correspond and we have labelled the edges with their corresponding p-modular irreducibles. From the Brauer tree one reads off the PIMs

$$V_1 \cdot V_{p-2} \cdot V_1 \quad \text{and} \quad V_{p-2} \cdot (V_1 \oplus V_3) \cdot V_{p-2}$$

for all primes $p \geqslant 7$ with $p \equiv 3 \pmod 4$. (In fact, these PIM structures are valid for all primes $p \geqslant 5$.) Note that the V_i and all of the PIMs for $\mathrm{L}_2(p)$ (and also $\mathrm{SL}_2(p)$) can be realised over $\mathbb{F}_p$.

Let W be the $\mathbb{F}_p\mathrm{L}_2(p)$-module $(V_1 \oplus V_3) \cdot V_{p-2}$ (with simple head). So W is a quotient of the PIM $V_{p-2} \cdot (V_1 \oplus V_3) \cdot V_{p-2}$ and therefore is unique. One can also read off from the PIMs that W has zero 1-cohomology whenever $p \geqslant 7$.

For p prime and $p \equiv 7 \pmod 8$, we define J_p to be $J_p \cong (2^{(p+1)/2} \times p^{p+2}){:}\mathrm{L}_2(p)$, in which the complementary $\mathrm{L}_2(p)$ act on $\mathrm{O}_p(J_p)$ as the module $W \cong (1 \oplus V_3) \cdot V_{p-2}$ and on $\mathrm{O}_2(J_p)$ as the module $1 \cdot U_1$ of Lemma 5. The groups $J_p/\mathrm{O}_p(J_p)$ are isomorphic to the groups M_p we constructed in [**1**].

3. *Perfect groups*

In this section, we construct infinite series of finite perfect groups which prove Theorems 1 and 3. We let $r \geqslant 11$ be a prime, and define G to be the direct product of certain perfect groups B_p for each prime p between 3 and r inclusive:

$$G = \prod_{p \in \pi} B_p = \prod_{p=3,\ p \text{ prime}}^{r} B_p,$$

where π is the set of odd primes not exceeding r. Firstly, we take $B_3 \cong 3^6{:}\mathrm{M}_{11}$, where $\mathrm{O}_3(B_3)$ when regarded as an $\mathbb{F}_3\mathrm{M}_{11}$-module is a uniserial module of shape $1 \cdot 5a$ (this module is isomorphic to the unique 6-dimensional submodule of the $\mathbb{F}_3$-permutation module of M_{11} on the 12 cosets of $\mathrm{L}_2(11)$). Note also that the composition factor $5a$ is absolutely irreducible. We have:

Lemma 6. *The $\mathbb{F}_3\mathrm{M}_{11}$-module $1 \cdot 5a$ has zero 1-cohomology.*

Proof. This is an easy calculation using Magma [**2**]. Alternatively, the trivial module has zero 1-cohomology since M_{11} is perfect, and it can be shown that the module $5a$ does not occur in the second Loewy layer of the trivial PIM. Thus both composition factors of $1 \cdot 5a$ have trivial 1-cohomology, and so does the whole module. □

Lemma 7. *If $p = 3$, so that $B_p = B_3 \cong 3^6{:}\mathrm{M}_{11}$, then $B_p = B_3$ has outer automorphism group of order 2. Thus $|\mathrm{Aut}\, B_p| = \frac{2}{3}|B_p| = \frac{p-1}{p}|B_p|$.*

Proof. Since $\mathrm{O}_3(B_3)$ is a characteristic subgroup of B_3, any automorphism of B_3 permutes the complements to $\mathrm{O}_3(B_3)$ in B_3. Now let S denote a complementary M_{11} in B_3. Since we have ensured that the $\mathbb{F}_3\mathrm{M}_{11}$-module $\mathrm{O}_3(B_3)$ has zero 1-cohomology, we may assume our automorphism, α say, normalises S. But M_{11} has trivial outer automorphism group, and adjusting α by an inner automorphism that is conjugation by an element of S, we may assume that α centralises S. So now α is an $\mathbb{F}_3S$-module automorphism of $\mathrm{O}_3(B_3) \cong 1 \cdot 5a$, and is thus a non-zero scalar. There are two of these and so $|\mathrm{Out}\, B_p| = 2$. In fact, $\mathrm{Aut}\, B_p \cong 3^5{:}(\mathrm{M}_{11} \times 2)$. □

For $p \geqslant 5$ and $p \not\equiv 7 \pmod 8$, we take $B_p \cong p_+^{1+2}{:}\mathrm{SL}_2(p)$. For $p \geqslant 5$ and $p \equiv 7 \pmod 8$, we take $B_p \cong p_+^{1+2}{:}\mathrm{SL}_2(p)$ or $B_p \cong J_p \cong (2^{(p+1)/2} \times p^{p+2}){:}\mathrm{L}_2(p)$, the group we constructed in Section 2 (we are free to choose either; this choice is necessary in order to prove Theorem 3).

The group $p_+^{1+2}{:}\mathrm{SL}_2(p)$ has a centre of order p, and is isomorphic to a vector stabiliser in the natural representation of $\mathrm{Sp}_4(p)$.

Lemma 8. *Let $H = N{:}K$ and let K act faithfully on N. Suppose $\alpha \in \operatorname{Aut} H$ centralises N and normalises K. Then α centralises K. (So $\alpha = 1$.)*

Proof. For all $g \in N$, $k \in K$ we have $(g^k)\alpha = g^k$ since $g^k \in N$. On the other hand $(g^k)\alpha = (g\alpha)^{(k\alpha)} = g^{(k\alpha)}$. So for all $g \in N$, $k \in K$ we have $g^{k(k\alpha)^{-1}} = g$, whence $k\alpha = k$ since K acts faithfully on N. □

Lemma 9. *For all primes $p \geqslant 5$ we have $\operatorname{Aut}(p_+^{1+2}{:}\mathrm{SL}_2(p)) \cong p^2{:}\mathrm{GL}_2(p)$. So if $B_p \cong p_+^{1+2}{:}\mathrm{SL}_2(p)$ we have $|\operatorname{Aut} B_p| = \frac{p-1}{p}|B_p|$.*

Proof. The 1-space stabiliser in $\mathrm{Sp}_4(p)$ is a group $p_+^{1+2}{:}\mathrm{GL}_2(p)$ which induces a group $p^2{:}\mathrm{GL}_2(p)$ of automorphisms on its normal subgroup $p_+^{1+2}{:}\mathrm{SL}_2(p)$.

The group $p_+^{1+2}{:}\mathrm{SL}_2(p)$ contains exactly p^2 involutions, which are permuted faithfully by the above group $p^2{:}\mathrm{GL}_2(p)$. Each of these involutions has centraliser of shape $p \times \mathrm{SL}_2(p)$, and these define the p^2 complements $\mathrm{SL}_2(p)$ [by taking the O^p or the derived subgroup]. Moreover, these involutions generate $p_+^{1+2}{:}2$, and support a natural affine plane structure; three involutions are collinear in this affine plane if and only if they generate a subgroup isomorphic to D_{2p}.

We already see the full automorphism group $p^2{:}\mathrm{GL}_2(p)$ of this affine plane, so the only way the automorphism group of $p_+^{1+2}{:}\mathrm{SL}_2(p)$ could be any bigger is if there were a non-trivial kernel, i.e. an automorphism centralising all p^2 involutions. Such an automorphism would have to normalise, and therefore by Lemma 8 centralise, each of the complements, as well as centralising the group $p_+^{1+2}{:}2$ generated by the involutions. Therefore it is the trivial automorphism on each complementary $\mathrm{SL}_2(p)$, and hence on $p_+^{1+2}{:}\mathrm{SL}_2(p)$, and the lemma follows. □

Lemma 10. *If $p \equiv 7 \pmod 8$ and H is the group $J_p \cong (2^{(p+1)/2} \times p^{p+2}){:}\mathrm{L}_2(p)$ we constructed in Section 2, then $\operatorname{Aut} H \cong (2^{(p-1)/2} \times p^{p+1}){:}(\mathrm{L}_2(p) \times \mathrm{C}_{p-1})$. So if $B_p \cong J_p$ we have $|\operatorname{Aut} B_p| = \frac{1}{2}\frac{p-1}{p}|B_p|$.*

Proof. The elementary abelian subgroups $\mathrm{O}_2(H)$ and $\mathrm{O}_p(H)$ are characteristic in H; therefore $K := \mathrm{O}_2(H) \times \mathrm{O}_p(H)$ is also characteristic in H. Since $\mathrm{O}_2(H)$ and $\mathrm{O}_p(H)$ both have zero 1-cohomology as $\mathrm{L}_2(p)$-modules (see Section 2), H has just one conjugacy class of complementary subgroups $\mathrm{L}_2(p)$. So let $\alpha \in \operatorname{Aut} H$ and let S be a complementary subgroup $\mathrm{L}_2(p)$. Modulo inner automorphisms, α normalises S. Now $\mathrm{O}_2(H)$ when regarded as an $\mathbb{F}_2S$-module does not admit the non-trivial outer automorphism of $S \cong \mathrm{L}_2(p)$, see Lemma 5. So α induces an inner automorphism when restricted to S, and adjusting by an inner automorphism of H that is conjugation by an element of S, we may assume that α centralises S. So now α induces an $\mathbb{F}_2S$-module automorphism on $\mathrm{O}_2(H)$ and an $\mathbb{F}_pS$-module automorphism on $\mathrm{O}_p(H)$, and both of these are scalars. Since $H \cong J_p$ has centre of order $2p$, the result follows. □

Lemma 11. *For all primes p with $3 \leqslant p \leqslant r$, the groups B_p are characteristic in G.*

Proof. Let π be the set of all primes between 3 and r inclusive. Let $N = \mathrm{F}(G)$, the Fitting subgroup of G, so that N is characteristic in G. Then

$$G/N \cong \prod_{p\in\pi} S_p,$$

where $S_3 \cong \mathrm{M}_{11}$ and $S_p \cong \mathrm{L}_2(p)$ whenever $p \geqslant 5$. So G/N has a unique normal subgroup N_p/N such that $N_p/N \cong S_p$, and for all p we get that N_p is characteristic in G. In fact

$$N_p = B_p \times \prod_{q\in\pi'} \mathrm{O}_{\{2,q\}}(B_q),$$

where $\pi' = \pi \setminus \{p\}$, with the $\mathrm{O}_{\{2,q\}}(B_q)$ being nilpotent groups of class at most 2. Therefore $B_p = N_p''$ is a characteristic subgroup of G. □

Since all of the B_p are characteristic in G, we have $\mathrm{Aut}\, G \cong \prod_{p\in\pi} \mathrm{Aut}\, B_p$. We have also established for all $p \in \pi$ that $|\mathrm{Aut}\, B_p| = \frac{p-1}{p}|B_p|$ or $\frac{1}{2}\frac{p-1}{p}|B_p|$, with the latter case occurring if and only if $B_p \cong J_p$. Therefore we have

$$\frac{|\mathrm{Aut}\, G|}{|G|} = \frac{1}{2^m} \times \prod_{p\in\pi} \frac{p-1}{p},$$

where m is the number of primes $p > 3$ for which $B_p \cong J_p$. Now when $p > 3$ the largest prime divisor of $|B_p|$ is p, and the primes dividing $|B_3|$ are 2, 3, 5 and 11. But $r \geqslant 11$, so the primes dividing $|G|$ are precisely the primes that do not exceed r, and thus we have

$$|\mathrm{Aut}\, G| = \frac{1}{2^{m-1}} \times \frac{1}{2} \times \prod_{p\in\pi} \frac{p-1}{p} \times |G| = \frac{1}{2^{m-1}} \times \phi(|G|).$$

When $m = 1$ this gives $|\mathrm{Aut}\, G| = \phi(|G|)$. We now invoke Dirichlet's Theorem that there are infinitely many primes p with $p \equiv 7 \pmod 8$ to complete the proofs of Theorems 1 and 3.

Remark. It is convenient but not essential to take all odd primes up to r in the definition of G. But every odd prime dividing $|G|$ must be one of these defining primes. To this end, let π be a set of odd primes such that $3, 5, 11 \in \pi$ and if $p \in \pi$ then $q \in \pi$ whenever q is an odd prime factor of $p-1$ or $p+1$. Let the B_p be as above. Then the group

$$G = \prod_{p\in\pi} B_p$$

satisfies $|\mathrm{Aut}\, G| = 2^{1-m}\phi(|G|)$ where m is the number of $p \in \pi$ such that $B_p \cong (2^{(p+1)/2} \times p^{p+2}){:}\mathrm{L}_2(p)$.

4. *Soluble groups*

In this section, we construct infinite series of finite soluble groups in order to prove Theorems 2 and 4.

We define B_3 to be the unique group of shape $3_+^{1+2}{:}4$ with centre of order 3; this group is `SmallGroup(108,15)` in various versions of Magma [**2**], including Version 2.10. Let π be a finite non-empty set of primes such that $p \equiv 1$ or $7 \pmod 8$ for all

$p \in \pi$, and let $\pi' = \pi \cup \{3\}$. We shall consider the groups:

$$G \;=\; \prod_{p\in\pi'} B_p \;\cong\; \left(\prod_{p\in\pi} p_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^-\right) \times 3_+^{1+2}{:}4,$$

where $B_p \cong p_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^-$ for all $p \in \pi$ and $2^{\cdot}\mathrm{S}_4^-$ denotes the proper double cover of S_4 in which transpositions do not lift to involutions.

It is well-known, see Dickson [**4**], that $\mathrm{L}_2(p)$ contains subgroups isomorphic to S_4 if and only if $p \equiv \pm 1 \pmod 8$. Such a subgroup is unique up to conjugacy in $\mathrm{PGL}_2(p)$, and is also self-normalising in $\mathrm{PGL}_2(p)$. Now the double cover $\mathrm{SL}_2(p)$ of $\mathrm{L}_2(p)$ contains just one involution, namely its central one, and so the preimage $2.\mathrm{S}_4$ in $\mathrm{SL}_2(p)$ is $2^{\cdot}\mathrm{S}_4^-$. Thus the group $B_p \cong p_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^-$ exists and has centre of order p. Note that the primes dividing $|G|$ are precisely those in the set $\{2,3\} \cup \pi$.

Lemma 12. *For all $p \in \pi$, the groups B_p are characteristic in G.*

Proof. Let $\pi'' := (\pi \cup \{3\}) \setminus \{p\} = \pi' \setminus \{p\}$. The group $\mathrm{O}_{2'}(G)$ is the direct product of the subgroups $W_p := \mathrm{O}_p(B_p) \cong p_+^{1+2}$. We have

$$\mathrm{O}_{\{2,p\}'}(G) \;=\; \mathrm{O}_{p'}(\mathrm{O}_{2'}(G)) \;=\; \prod_{q\in\pi''} W_q.$$

Thus we calculate that

$$H \;:=\; \mathrm{C}_G(\mathrm{O}_{p'}(\mathrm{O}_{2'}(G))) \;=\; B_p \times \prod_{q\in\pi''} \mathrm{Z}(W_q) \;\cong\; p_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^- \times \prod_{q\in\pi''} \mathrm{C}_q.$$

Now B_p is characteristic in H, since it is the subgroup generated by the elements of order 4. Since H is a characteristic subgroup of G we conclude that B_p is also. □

Lemma 13. *The group B_3 is also characteristic in G.*

Proof. This proof is very similar to the proof of Lemma 12, and we use the notation W_q from the proof of that lemma here. The group

$$H \;:=\; \mathrm{C}_G(\mathrm{O}_{3'}(\mathrm{O}_{2'}(G))) \;=\; B_3 \times \prod_{q\in\pi} \mathrm{Z}(W_q) \;\cong\; 3_+^{1+2}{:}4 \times \prod_{q\in\pi} \mathrm{C}_q$$

is a characteristic subgroup of G. Now B_p is characteristic in H, since it is the subgroup generated by the elements of order 4. Since H is a characteristic subgroup of G we conclude that B_p is also. □

Lemma 14. *For all $p \in \pi$, we have* $\mathrm{Aut}\, B_p \cong p^2{:}(2^{\cdot}\mathrm{S}_4^- \circ \mathrm{C}_{p-1})$. *So for all such primes p we have* $|\mathrm{Aut}\, B_p| = \frac{1}{2}\frac{p-1}{p}|B_p|$.

Proof. We adapt the proof of Lemma 9. First note that $H = B_p \cong p_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^-$ embeds in $p_+^{1+2}{:}\mathrm{GL}_2(p)$, in which its normaliser is $p_+^{1+2}{:}(2^{\cdot}\mathrm{S}_4^- \circ \mathrm{C}_{p-1})$, and therefore its automorphism group contains $p^2{:}(2^{\cdot}\mathrm{S}_4^- \circ \mathrm{C}_{p-1})$.

Note that the latter group acts transitively and faithfully on the p^2 involutions in H, so if we have any further automorphism α, we may assume α fixes one of these involutions, say z. Therefore α fixes $\mathrm{C}_H(z) \cong p \times 2^{\cdot}\mathrm{S}_4^-$, and therefore normalises the corresponding subgroup $K = \mathrm{O}^p(\mathrm{C}_H(z)) \cong 2^{\cdot}\mathrm{S}_4^-$ of H.

Now both α and K act on the affine plane defined by the p^2 involutions (as in

Lemma 9), and both fix the same point, which we can regard as the origin. In the resulting action on the vector space of order p^2, the image of α normalises the image of K inside $\mathrm{GL}_2(p)$ (indeed, $K{:}\langle\alpha\rangle$ acts on this vector space). But this action of K is faithful and $\mathrm{N}_{\mathrm{GL}_2(p)}(2^{\cdot}\mathrm{S}_4^-) \cong 2^{\cdot}\mathrm{S}_4^- \circ \mathrm{C}_{p-1}$. But we have already seen a group $p^2{:}(2^{\cdot}\mathrm{S}_4^- \circ \mathrm{C}_{p-1})$ of automorphisms of H acting faithfully on the affine plane, and so we may assume that α acts trivially on the affine plane. In other words, α centralises all p^2 involutions in H, so centralises the group $p_+^{1+2}{:}2$ which they generate.

We now know that α centralises p_+^{1+2}, and normalises a complementary $2^{\cdot}\mathrm{S}_4^-$. Therefore, by Lemma 8, α is the trivial automorphism of H. This completes the proof of the lemma. □

An easy calculation gives $\mathrm{Aut}\, B_3 \cong 3^2{:}\mathrm{SD}_{16}$. Since all of the B_p are characteristic in G, we obtain $\mathrm{Aut}\, G \cong \prod_{p\in\pi\cup\{3\}} \mathrm{Aut}\, B_p$. Therefore we have

$$\frac{|\mathrm{Aut}\, G|}{|G|} = \frac{4}{3} \times \left(\frac{1}{2}\right)^{|\pi|} \times \prod_{p\in\pi} \frac{p-1}{p} = \frac{1}{2^{|\pi|-2}} \times \frac{\phi(|G|)}{|G|},$$

and so

$$\frac{|\mathrm{Aut}\, G|}{\phi(|G|)} = \frac{1}{2^{|\pi|-2}}.$$

When $|\pi| = 2$ this gives $|\mathrm{Aut}\, G| = \phi(|G|)$. Dirichlet's Theorem tells us that π can be made arbitrarily large, thus proving Theorem 2, and also that there are infinitely many size 2 possibilities for π, thus proving Theorem 4.

Remark. In the above construction for G we can replace the group B_3 by the cyclic group of order 3, in which case $|\mathrm{Aut}\, G| = 2^{1-|\pi|}\phi(|G|)$. However the proofs are slightly different. The smallest non-cyclic group G we know of that satisfies $|\mathrm{Aut}\, G| = \phi(|G|)$ is now the group $G \cong 3 \times 7_+^{1+2}{:}2^{\cdot}\mathrm{S}_4^-$ of order 49392, narrowly beating the example $2^4{:}\mathrm{L}_3(2) \times 3 \times 7$ of order 56448 that we gave in [**1**].

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References

1. J. N. Bray and R. A. Wilson. On the orders of automorphism groups of finite groups. *Bull. London Math. Soc.*, to appear.

2. J. J. Cannon *et al*. The Magma programming language, Version 2.10. School of Mathematics and Statistics, University of Sydney (2003).

3. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson. An ATLAS of Finite Groups. *Clarendon Press, Oxford* (1985; reprinted with corrections, 2003).

4. L. E. Dickson. Linear Groups, with an Exposition of the Galois Field Theory. *Teubner, Leipzig* (1901), *reprinted Dover, New York* (1958).

5. E. I. Khukhro and V. D. Mazurov (Eds). Unsolved problems in group theory. The Kourovka Notebook, no. 15. Novosibirsk, 2002.

6. E. Schenkman. The existence of outer automorphisms of some nilpotent groups of class 2. *Proc. Amer. Math. Soc.* **6** (1955), 6–11.

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