

MTH4103: Geometry I



Mid-term Test 2014 Solutions

Duration: 40 minutes

Date and Time: 19th February 2014, in the hour 11:00–12:00

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The duration of this test is **40 minutes**. Answer **all** the questions. The marks (out of 100) allocated to each question are shown after the question numbers. For most questions, only the final answer to a question will be marked, so indicate this answer clearly. Simplify your answers as much as possible.

Calculators are **not** permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Answer all questions in the spaces provided. You may do additional rough work on the backs of the question sheets or at the end, but this will not be looked at.

Do **not** turn over until an invigilator instructs you to do so.

For Examiner's use only

Question	Mark	Question	Mark
1	10	5	20
2	10	6	20
3	10	7	10
4	10	8	10
		Total (%)	100

Question 1 (10 marks). Let $\mathbf{a} = \begin{pmatrix} 4 \\ -7 \\ -4 \end{pmatrix}$. Determine $|\mathbf{a}|$.

Solution. We have $|\mathbf{a}| = \sqrt{4^2 + (-7)^2 + (-4)^2} = \sqrt{16 + 49 + 16} = \sqrt{81} = 9$. ✓

The marking scheme was as follows: 10 marks for the answer “9”; 5 marks for “ $\sqrt{81}$ ”; 0 marks for any other answer, including “ ± 9 ”.

Question 2 (10 marks). Determine parametric equations for the line ℓ through the points $A = (-1, -4, 3)$ and $B = (0, -1, 2)$.

Solution. Let $A = (-1, -4, 3)$ and $B = (0, -1, 2)$ have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Then the line ℓ through A and B has direction

$$\mathbf{u} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix},$$

and so has parametric equations

$$\left. \begin{array}{l} x = -1 + \lambda \\ y = -4 + 3\lambda \\ z = 3 - \lambda \end{array} \right\} \text{.} \quad \checkmark$$

(Here, it is valid to take any nonzero scalar multiple of $\mathbf{u}$ as ‘the’ direction vector. In particular, it is perfectly valid to use $-\mathbf{u} = \mathbf{a} - \mathbf{b}$ for this, which would negate coefficients of λ seen above. Changing our ‘favourite’ point on the line [we are using A here] gives further different solutions.)

Lose 2 marks if the brace is omitted. No marks if you have not written down parametric equations, or if your answer is ambiguous.

Question 3 (10 marks). Consider the following system of linear equations defined over $\mathbb{R}$ in variables x , y and z :

$$\left. \begin{array}{l} 3y + 2z = -1 \\ -3x - 2y + 3z = -2 \\ x + 4y + 9z = -4 \end{array} \right\}.$$

State clearly, and in order, **all** operations that must be performed in Step 1 of the Gaußian elimination process (as defined in this module) to bring the above system of equations into echelon form. [You do not have to perform these operations, except if you have trouble visualising the operations you have to do. Nor will the system of equations be in echelon form at the end of Step 1.]

Solution. The (two) operations are as follows.

1. Interchange (swap) Equations 1 and 2.
2. Add $\frac{1}{3}$ times Equation 1 to Equation 3. ✓

(Recall that Step 1 is concerned with eliminating the variable x (or x_1) from all equations except possibly the first.)

Please note that you were **not** asked to perform these operations. You were certainly not asked to solve the equations (the solution is $x = \frac{8}{35}$, $y = -\frac{1}{35}$ and $z = -\frac{16}{35}$). You were not even asked to echelonise the system of equations. You were not even asked to give the final operation (add $-\frac{10}{9}$ times Equation 2 to Equation 3) to finally bring the system into echelon form, as that operation belongs to Step 2. (I was very generous in ignoring operations in Step 2, whether right or wrong.) Also, writing down the description of Step 1 in my notes does not constitute a valid answer to this question.

Question 4 (10 marks). Define carefully, but **without using coördinates**, the *scalar product* (or *dot product*) $\mathbf{u} \cdot \mathbf{v}$ of two vectors $\mathbf{u}$ and $\mathbf{v}$.

Solution. If $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ then we define $\mathbf{u} \cdot \mathbf{v} := 0$. Else $\mathbf{u} \cdot \mathbf{v} := |\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. ✓

This question was answered extremely poorly, with most of you scoring zero (see the marking scheme below). In particular, the cases $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ were rarely mentioned, which is vital for a proper definition of $\mathbf{u} \cdot \mathbf{v}$. Also, “ $\mathbf{u} \cdot \mathbf{v} := |\mathbf{u}||\mathbf{v}| \cos \theta$ ” introduces a hitherto undefined entity, namely θ , so you should specify what θ is, rather than hoping that the reader would guess. Lose 5 marks if the cases $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$ are omitted, or if θ is not defined. (So lose 10 marks if both of these infractions occur.)

Question 5 (20 marks).

- (a) **(7 marks)** Given two planes in $\mathbb{R}^3$, describe (geometrically) their possible intersections. Also say which (case or) cases are typical, and which are exceptional.
- (b) **(3 marks)** Given three planes in $\mathbb{R}^3$, what do we expect their intersection to be in general? Ignore exceptional cases.
- (c) **(10 marks)** Planes Π_1 and Π_2 , with Cartesian equations $3x + y - z = 3$ and $2y - z = 4$ respectively, intersect in a line ℓ . Find a vector equation for ℓ .

Solution.

- (a) The possible cases are as follows.

- (i) A line (typical).
- (ii) The empty set (atypical).
- (iii) A plane ([very] atypical). ✓

The intended marking scheme was 2 marks per case, with a bonus mark if all cases were correct. In Case (i) the planes are not parallel, whereas in Cases (ii) and (iii) they are, with Case (ii) being typical and Case (iii) atypical when the planes are parallel. Also, please do not write “the planes have no intersection” to describe Case (ii): it is incredibly painful (and wrong).

- (b) A point. ✓

- (c) We observe that the system of equations

$$\left. \begin{array}{l} 3x + y - z = 3 \\ 2y - z = 4 \end{array} \right\}.$$

specifying the intersection of Π_1 and Π_2 is already in echelon form. Solving by back substitution gives us $z = t$, where t can be any real number. So now $2y = 4 + z$, giving us $y = 2 + (\frac{1}{2})t$, and finally $3x = 3 + z - y = 3 + t - 2 - (\frac{1}{2})t$, which gives us $x = \frac{1}{3} + (\frac{1}{6})t$. So possible vector equations for ℓ include

$$\mathbf{r} = \begin{pmatrix} 1/3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1/6 \\ 1/2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} 1/3 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}. \quad \checkmark$$

(The first version is what we naturally get, and the second is obtained by multiplying the direction vector in the first by 6. Direction vectors for ℓ can be obtained by taking the cross product of normal vectors to the two planes, but this was not an intended method of solution, partly because we still need to be able to locate a point on ℓ .)

If you did not perform back substitution as detailed in the notes, you may have obtained a different vector on ℓ . As I did not specify that you use back substitution, these answers were allowed if they defined ℓ . But if you fail to use the usual methods you risk your marker failing to spot correct answers.)

Question 6 (20 marks). A regular tetrahedron in $\mathbb{R}^3$ can be taken to have vertices at $O = (0, 0, 0)$, $A = (0, 1, 1)$, $B = (1, 0, 1)$ and $C = (1, 1, 0)$, which have position vectors $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.

- (a) **(7 marks)** Calculate the vector product $\mathbf{a} \times \mathbf{b}$.
- (b) **(7 marks)** Calculate the volume of the tetrahedron (this is $\frac{1}{6} \times$ the volume of a parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$).
- (c) **(6 marks)** Calculate $\cos \theta$, where θ is the angle between $\mathbf{c} - 2\mathbf{a}$ and the vector represented by $\overrightarrow{AB}$.

Solution.

$$(a) \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}. \checkmark$$

(It is recommended to check that your answer to $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$; some answers I received did not have this property. Worse still, some answers I received were not even vectors.)

- (b) The volume is $\frac{1}{6} |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \frac{1}{6} |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| = \frac{1}{6} |1 + 1 + 0| = \frac{1}{3}. \checkmark$ (I used Theorem 6.6 here. If you calculated $\mathbf{b} \times \mathbf{c}$ [which is the position vector of $(-1, 1, 1)$], you probably took longer but should have obtained the same answer.)

$$(c) \text{ We have } \mathbf{c} - 2\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \text{ and } \mathbf{b} - \mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \text{ where } \mathbf{b} - \mathbf{a} \text{ is the vector represented by } \overrightarrow{AB}. \text{ So } |\mathbf{c} - 2\mathbf{a}| = \sqrt{6}, |\mathbf{b} - \mathbf{a}| = \sqrt{2} \text{ and } (\mathbf{c} - 2\mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = 1 + 1 + 0 = 2. \text{ Therefore } \cos \theta = \frac{2}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{3}}. \checkmark$$

Question 7 (10 marks). Exactly which of the following statements are true, and exactly which are false? [A statement of the form “ s_1 and s_2 are true” will *not* be taken to mean that you think s_3 , s_4 and s_5 are false.]

- (a) If $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$ are collinear.
- (b) If $\mathbf{u} \cdot \mathbf{v} = 0$ for all vectors $\mathbf{v}$ then $\mathbf{u} = \mathbf{0}$.
- (c) If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
- (d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ for all vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.
- (e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ for all nonzero vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Solution. (a), (b) and (d) are true, while (c) and (e) are false. ✓ The solutions you produced for this were little better than using coin tosses with a fair coin to decide your answers. Marking scheme: 10 marks if all correct; 3 marks if 4 correct; otherwise (including cases of ambiguity) 0 marks. Reasons (not required) were as follows.

- (a) If $\mathbf{w} = \mathbf{0}$ or $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then we are fine. Else, up to oppositeness, there is a unique direction orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Both $\mathbf{w}$ and $\mathbf{u} \times \mathbf{v}$ must have one of these directions.
- (b) Part (c) of Feedback Question on Coursework 2.
- (c) Counterexample $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$ (among many others).
- (d) Uses distributive and anti-commutative laws (Theorems 6.8 and 6.2 in notes).
- (e) Counterexample $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = (\mathbf{i} \times \mathbf{j}) \times \mathbf{k} = \mathbf{0}$ (actually done in lectures). We can also take $\mathbf{u} = \mathbf{v} = \mathbf{w} \neq \mathbf{0}$.

Question 8 (10 marks). Pick a plane Π passing through the points $A = (1, 2, 3)$ and $B = (4, 5, 6)$ with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. (There are infinitely many such planes to choose from.) Write down a Cartesian equation for Π .

Solution. (NB: Answer not unique.) A specific answer was required, and one possible answer is $2x - y - z = -3$. ✓ In order to find the most general answer, we let $\mathbf{n} \neq \mathbf{0}$ be a vector normal to Π , and let $\mathbf{u}$ be the vector represented by $\overrightarrow{AB}$. We require that

$\mathbf{n} \cdot \mathbf{u} = 0$. Since $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$ and $\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$, we require that

$3a + 3b + 3c = 0$, and so $a + b + c = 0$. (Also note that $(a, b, c) \neq (0, 0, 0)$ since $\mathbf{n} \neq \mathbf{0}$.) So Π has vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where d is determined by forcing $(1, 2, 3)$ to lie on Π , and this gives $d = a + 2b + 3c$.

For ease of marking, I subtracted 0 in the form of $2(a + b + c)$ from d and observed that $d = c - a$. This is tantamount to observing that $(-1, 0, 1)$ lies on Π , which it most certainly does, as it lies on the line determined by A and B .

So the most general possibility for Π has a Cartesian equation $ax + by + cz = c - a$ for some $a, b, c \in \mathbb{R}$ such that $(a, b, c) \neq (0, 0, 0)$ and $a + b + c = 0$.