MAS400: Solutions 8

All questions have the following. Let $v_1, \dots, v_n$ be linearly independent vectors in $\mathbb{R}^n$, such that $(v_1, \dots, v_n)$ has Gram–Schmidt orthogonalisation $((v_1^*, \dots, v_n^*), (\mu_{st}))$. Determine the Gram–Schmidt orthogonalisations $((w_1^*, \dots, w_n^*), (\xi_{st}))$ of $(w_1, \dots, w_n)$ in the following cases. Throughout, I shall assume well-known properties of the GSO such as $\langle v_1^*, \dots, v_k^* \rangle_{\mathbb{R}} = \langle v_1, \dots, v_k \rangle_{\mathbb{R}}$ for all k, and that v_k^* is the (orthogonal) projection of v_k onto $\langle v_1, \dots, v_{k-1} \rangle_{\mathbb{R}}^{\perp}$ for all k.

Throughout, let $V_k = \langle v_1, \dots, v_k \rangle = \langle v_1^*, \dots, v_k^* \rangle$ and $W_k = \langle w_1, \dots, w_k \rangle$, for all k with $0 \leqslant k \leqslant n$.

1. Case $w_i = \lambda v_i$, $w_j = v_j$ if $j \neq i$, where $\lambda \neq 0$ $(\lambda \in \mathbb{R})$.
   In this case $V_k = W_k$ for all k, including the case $k = i$. Since $w_k = v_k$ if $k \neq i$ then clearly $w_k^* = v_k^*$ for such k. Now w_i^* is the orthogonal projection of $w_i = \lambda v_i$ onto $W_{i-1}^{\perp} = V_{i-1}^{\perp}$, and so $w_i^* = \lambda v_i^*$. This deals with the GSO basis. The only pairs (s,t) where ξ_{st} might differ from μ_{st} are those for which $s > t$ and where $w_s \neq v_s$ or $w_t^* \neq v_t^*$. Thus $\xi_{st} = \mu_{st}$ unless $i = s > t$ or $s > t = i$. In the case $i = s > t$ we have $\xi_{st} = \xi_{it} = \frac{w_i \cdot w_t^*}{w_t^* \cdot w_t^*} = \frac{(\lambda v_i) \cdot v_t^*}{v_t^* \cdot v_t^*} = \lambda \frac{v_i \cdot v_t^*}{v_t^* \cdot v_t^*} = \lambda \mu_{it} = \lambda \mu_{st}$. In the case $s > t = i$ we have $\xi_{st} = \xi_{si} = \frac{w_s \cdot w_i^*}{w_i^* \cdot w_i^*} = \frac{v_s \cdot (\lambda v_i^*)}{(\lambda v_i^*) \cdot (\lambda v_i^*)} = \frac{1}{\lambda} \frac{v_s \cdot v_i^*}{v_i^* \cdot v_i^*} = (\frac{1}{\lambda}) \mu_{si} = (\frac{1}{\lambda}) \mu_{st}$.

2. Case $w_i = v_i - \lambda v_k$ with $k < i$, $w_j = v_j$ if $j \neq i$.
   But for all $k < i$ we have $w_k = v_k$ and thus $W_k = V_k$ for all $k < i$. Now $v_i - w_i = \lambda v_j \in V_j \subseteq V_{i-1}$, and so $W_i = V_i$ too. Since $w_k = v_k$ for $k > i$ we conclude that $W_k = V_k$ for all k. Firstly, let us consider the GSO bases. For all k, v_k^* is the projection of v_k onto $V_{k-1}^{\perp} \cap V_k$. Also w_k^* is the projection of w_k onto $W_{k-1}^{\perp} \cap W_k = V_{k-1}^{\perp} \cap V_k$, and since $w_k - v_k \in V_{k-1}$ for all k we get that $w_k^* = v_k^*$ for all k. If $k < l$ then $\xi_{kl} = \mu_{kl} = 0$ and if $k = l$ then $\xi_{kl} = \mu_{kl} = 1$. In the main case when $k > l$ we have $\xi_{kl} = \frac{w_k \cdot w_l^*}{w_l^* \cdot w_l^*} = \frac{w_k \cdot v_l^*}{v_l^* \cdot v_l^*}$. So if $k \neq i$ we have $w_k = v_k$ and thus $\xi_{kl} = \mu_{kl}$. If $k = i$ then $w_k = w_i = v_i - \lambda v_j$. Therefore we have $\xi_{il} = \frac{w_i \cdot v_l^*}{v_l^* \cdot v_l^*} = \frac{(v_i - \lambda v_j) \cdot v_l^*}{v_l^* \cdot v_l^*} = \frac{v_i \cdot v_l^*}{v_l^* \cdot v_l^*} - \lambda \frac{v_j \cdot v_l^*}{v_l^* \cdot v_l^*} = \mu_{il} - \lambda \frac{v_j \cdot v_l^*}{v_l^* \cdot v_l^*}$, which is $\mu_{il} - \lambda \mu_{jl}$ if $l < j$. Note that $v_k \cdot v_l^* = 0$ if $k < l$ as $v_k \in V_{l-1}$, which is perpendicular to v_l^*. Also $(v_k - v_k^*) \in V_{k-1}$, which is perpendicular to v_k^*, and so $v_k \cdot v_k^* = v_k^* \cdot v_k^*$ for all k. Thus $\xi_{ij} = \mu_{ij} - \lambda = \mu_{ij} - \lambda \mu_{jj}$ and $\xi_{il} = \mu_{il} = \mu_{il} - \lambda \mu_{jl}$ if $j < l < i$. Therefore $\xi_{il} = \mu_{il} - \lambda \mu_{jl}$ whenever $l < i$.

3. Case $w_i = v_{i+1}$, $w_{i+1} = v_i$, $w_j = v_j$ if $j \neq i, i+1$.
Firstly, let us consider the GSO bases. For all k, v_k^* is the projection of v_k onto $V_{k-1}^{\perp} \cap V_k$. If $k \neq i, i+1$ then $W_{k-1} = V_{k-1}$ and $W_k = V_k$ and so $w_k^* = v_k^*$ for all such k. Similarly $V_{i-1} = W_{i-1}$ and $V_{i+1} = W_{i+1}$ and so $U := \langle v_i^*, v_{i+1}^* \rangle = \langle w_i^*, w_{i+1}^* \rangle$. Now V_{i+1} is the direct sum of U and V_{i-1}. Thus considering the U-components of $v_i = w_{i+1}$ and $v_{i+1} = w_i$ gives the equations:

$$\begin{aligned} w_{i+1}^* + \xi_{i+1,i} w_i^* &= v_i^*, \\ w_i^* &= v_{i+1}^* + \mu_{i+1,i} v_i^*. \end{aligned}$$

Therefore, $w_{i+1}^* = v_i^* - \xi_{i+1,i} w_i^* = -\xi_{i+1,i} v_{i+1}^* + (1 - \xi_{i+1,i}\mu_{i+1,i}) v_i^*$. The inner product $0 = w_{i+1}^* \cdot w_i^*$ gives the equation:

$$\xi_{i+1,i}(v_{i+1}^* \cdot v_{i+1}^* + \mu_{i+1,i}\bar{\mu}_{i+1,i} v_i^* \cdot v_i^*) = \bar{\mu}_{i+1,i}(v_i^* \cdot v_i^*),$$

which allows one to determine $\xi_{i+1,i}$ in terms of the μs and v^*s. (Note that $v_{i+1}^* \cdot v_{i+1}^* + \mu_{i+1,i}\bar{\mu}_{i+1,i} v_i^* \cdot v_i^*$ is in fact $w_i^* \cdot w_i^*$, and is thus nonzero.) Thus w_{i+1}^* can be written in terms of the μs and v^*s.

Now we consider (most of) the ξ_{kl}. If $k = l$ then $\xi_{kl} = \mu_{kl} = 1$ and if $k < l$ then $\xi_{kl} = \mu_{kl} = 0$. Also if $k, l \notin \{i, i+1\}$ then the relevant vs and ws are equal and so $\xi_{kl} = \mu_{kl}$ in these cases too. If $k \leqslant i-1$ then $\xi_{i,k} = \frac{w_i \cdot w_k^*}{w_k^* \cdot w_k^*} = \frac{v_{i+1} \cdot v_k^*}{v_k^* \cdot v_k^*} = \mu_{i+1,k}$ and $\xi_{i+1,k} = \frac{w_{i+1} \cdot w_k^*}{w_k^* \cdot w_k^*} = \frac{v_i \cdot v_k^*}{v_k^* \cdot v_k^*} = \mu_{i,k}$. The remaining cases are nastier. (Recall that we have already calculated w_i^*, w_{i+1}^* and $\xi_{i+1,i}$ in terms of the μs and v^*s.) For all $k > i+1$ we have:

$$\begin{aligned} (w_i^* \cdot w_i^*)\xi_{ki} &= w_k \cdot w_i^* = v_k \cdot (v_{i+1}^* + \mu_{i+1,i} v_i^*) \\ &= v_k \cdot v_{i+1}^* + \bar{\mu}_{i+1,i}(v_k \cdot v_i^*) \\ &= (v_{i+1}^* \cdot v_{i+1}^*)\mu_{k,i+1} + \bar{\mu}_{i+1,i}(v_i^* \cdot v_i^*)\mu_{ki}, \end{aligned}$$

along with:

$$\begin{aligned} (w_{i+1}^* \cdot w_{i+1}^*)\xi_{k,i+1} &= w_k \cdot w_{i+1}^* = v_k \cdot (-\xi_{i+1,i} v_{i+1}^* + (1 - \xi_{i+1,i}\mu_{i+1,i}) v_i^*) \\ &= -\bar{\xi}_{i+1,i}(v_k \cdot v_{i+1}^*) + (1 - \bar{\xi}_{i+1,i}\bar{\mu}_{i+1,i})(v_k \cdot v_i^*) \\ &= -\bar{\xi}_{i+1,i}(v_{i+1}^* \cdot v_{i+1}^*)\mu_{k,i+1} + (1 - \bar{\xi}_{i+1,i}\bar{\mu}_{i+1,i})(v_i^* \cdot v_i^*)\mu_{ki}. \end{aligned}$$

We have already calculated that $\xi_{i+1,i} = \bar{\mu}_{i+1,i} \frac{v_i^* \cdot v_i^*}{w_i^* \cdot w_i^*}$ above.

I give only partial answers to the remaining questions, which (try to) detail what happens to the GSO when the remaining elementary row operations are applied to a basis. The ones relevant for `BasisReduction` are in Questions 2 and 3. In both the remaining cases, one is usually better off calculating the new GSO from scratch.

4. Case $w_i = v_i - \lambda v_k$ with $k > i$, $w_j = v_j$ if $j \neq i$.
   (This should be contrasted with the answer to Question 2.)

5. Case $w_i = v_k$, $w_k = v_i$ (wlog $i < k$), $w_j = v_j$ if $j \neq i, k$.
   Given the complications in the answer to Question 3, you can imagine how much worse the general case is. Clearly $W_j = V_j$ if $j < i$ or $j \geqslant k$. Thus $w_j^* = v_j^*$ if $j < i$ or $j > k$. Therefore $\xi_{jl} = \mu_{jl}$ whenever $j \leqslant l$ or ($l < i$ and $j \notin \{i, k\}$) or $j, l > k$. The rest is harder. We have that w_i^* is the projection of v_k^* onto $V_{i-1}^{\perp}$, and so we get

   $$w_i^* = v_k + \sum_{j=i}^{k-1} \mu_{kj} v_j^*.$$

   (Recall that $w_i = v_k = v_k^* + \sum_{j=1}^{k-1} \mu_{kj} v_j^*$.) For $i < j < k$ we get

   $$w_j = v_j = v_j^* + \sum_{l=1}^{j-1} \mu_{jl} v_l^*$$