MAS400: Solutions 4

Throughout this sheet F will denote an arbitrary field unless otherwise stated.

1. Let $f_1 = xy - 1$ and $f_2 = y^2 - 1$ be elements of $\mathbb{Q}[x, y]$. Show that $\{f_1, f_2\}$ is *not* a Gröbner basis for $\langle f_1, f_2\rangle$, with respect to $\leqslant_{\text{lex}}$ and $x > y$.

   If $\{f_1, f_2\}$ is a Gröbner basis then f rem $(f_1, f_2) = f$ rem (f_2, f_1) for all $f \in \mathbb{Q}[x, y]$. However, in lectures we saw that $x^2y+xy^2+y^2$ rem $(f_1, f_2) = x+y+1$, while $x^2y+xy^2+y^2$ rem $(f_2, f_1) = 2x+1$ (the latter calculation was left as an exercise).

   Alternatively: Newer technology (Theorem T) indicates we must consider whether $S(f_1, f_2)$ rem $(f_1, f_2) = 0$. We have $\operatorname{lcm}(\operatorname{lm} f_1, \operatorname{lm} f_2) = xy^2$, so that $S(f_1, f_2) = \frac{xy^2}{xy} f_1 - \frac{xy^2}{y^2} f_2 = yf_1 - xf_2 = -y + x$. Multivariate division of $S(f_1, f_2)$ by (f_1, f_2) or (f_2, f_1) gives $q_1 = q_2 = 0$ and remainder $x - y \neq 0$, regardless of the monomial order chosen, so that $\{f_1, f_2\}$ is not a Gröbner basis for $\langle f_1, f_2\rangle$.
   (Note that in this case it is fairly easy to spot a suitable linear combination $f = p_1f_1 + p_2f_2$ of f_1 and f_2 such that f rem $(f_1, f_2) \neq 0$; Theorem T and the S-polynomials just provides a systematic way of finding suitable linear combinations.)

2. Let $G = \{g_1, \ldots, g_t\} \subseteq F[x_1, \ldots, x_n]$ with $0 \notin G$. Suppose that, with respect to a fixed monomial order, we have:
   $$\forall f \in \langle g_1, \ldots, g_t\rangle, \quad f \text{ rem } (g_1, \ldots, g_t) = 0.$$
   Show that G is a Gröbner basis for $\langle G\rangle$.

   We consider the algorithm `MultivariateDivision`. On each iteration of the while-loop we may add $\operatorname{lt} p$ to r, or leave r unchanged. But each iteration of the while-loop also strictly decreases $\operatorname{mdeg} p$, so no term that is added to r can cancel previous terms. For $f \in \langle G\rangle$ the eventual remainder is zero, and so the else-part of the if-loop in the while-loop is never executed; therefore at all times during the execution of `MultivariateDivision` we have $p = 0$ or that $\operatorname{lt} p$ is divisible by (at least) one of the $\operatorname{lt} g_i$. If $f \neq 0$ then upon starting the first iteration of the while loop we have $p = f$, so that $\operatorname{lt} f$ is divisible by (at least) one of the $\operatorname{lt} g_i$, and thus $\operatorname{lt} f \in \langle \operatorname{lt} g_1, \ldots, \operatorname{lt} g_t\rangle$. Therefore $\langle \operatorname{lt}(\langle G\rangle)\rangle \leqslant \langle \operatorname{lt} g_1, \ldots, \operatorname{lt} g_t\rangle$, and so G is a Gröbner basis for $\langle G\rangle$.

3. Let R be a ring (commutative with 1). Show that each ideal of R is finitely generated if and only if there are no infinite ascending chains of ideals:

$$I_0 < I_1 < I_2 < I_3 < \cdots.$$

NB: I have proved a very similar result in lectures.

IF I is not finitely generated then we define $I_0 = 0$, and I_m for $m \geqslant 1$ recursively as follows: We pick $r_m \in I \setminus I_{m-1}$. Since $I_{m-1} = \langle r_1, \ldots, r_{m-1} \rangle$ is finitely generated and I is not, we can always pick such an r_m for all m. Then $I_0 < I_1 < I_2 < \cdots$ is a strictly ascending chain of R-ideals contained in I.

If each ideal is finitely generated we let $I_0 \leqslant I_1 \leqslant I_2 \leqslant \cdots$ be an ascending chain of ideals (possibly with equalities at various points). Now $I = \bigcup_{n=0}^{\infty} I_n$ is also an ideal (exercise), and so I is finitely generated, say $I = \langle r_1, \ldots, r_s \rangle$. For $i \in \{1, \ldots, s\}$ there exists $n_i \in \mathbb{N}$ such that $r_i \in I_{n_i}$. Let $N = \max(n_1, \ldots, n_s)$. For $n \geqslant N$ we have $I_n \leqslant I$ (since I is the union of the I_n) and $I \leqslant I_n$ (since I_n contains a generating set for I); thus $I_n = I$. Therefore $I_N = I_{N+1} = I_{N+2} = \cdots$, and no strictly infinite ascending chain of ideals exists.