MAS400: Exercises 3

Throughout this sheet F will denote an arbitrary field unless otherwise stated.

1. Perform multivariate division in $F[x, y]$ for the following cases. For critical cases, one may assume that $F = \mathbb{Q}$. Do them also with f_1 and f_2 interchanged, and for the monomial orders $\leqslant_{\text{lex}}$ and $\leqslant_{\text{grlex}}$ in which $x > y$. (You can also see what happens if $x < y$ if you like.)

   (a) $f = xy^3 - xy^2$, $f_1 = xy - x$, $f_2 = y^3$.

   (b) $f = x^2 + xy^3$, $f_1 = xy + 1$, $f_2 = y^2$.

2. Perform multivariate division in $F[x]$ when $f \in \{x^4, x^4 - 1\}$, $f_1 = x^3 + 1$, $f_2 = x^2 + 1$, and also with f_1 and f_2 interchanged. Note that there is a unique monomial order in this case. Although $F[x]$ is univariate, we cannot use univariate division as we are trying to simultaneously divide by two polynomials. What is the ideal $I = \langle f_1, f_2 \rangle$ anyway?

3. Let $I = \langle \underline{x}^{\alpha_1}, \underline{x}^{\alpha_2}, \ldots, \underline{x}^{\alpha_s} \rangle$ be a (finitely generated) monomial ideal of $F[x_1, \ldots, x_n]$, and $f \in F[x_1, \ldots, x_n]$. Show that if we apply multivariate division to divide f by $(\underline{x}^{\alpha_1}, \underline{x}^{\alpha_2}, \ldots, \underline{x}^{\alpha_s})$ then we get remainder $r = 0$ if and only if $f \in I$ (regardless of the fixed monomial order used).