MAS400: Exercises 1

Throughout this sheet F will denote an arbitrary field unless otherwise stated.

1. For the following pairs of elements $f, g \in F[x]$ find the g.c.d. (h say) of f and g and also find elements $a, b \in F[x]$ such that $af + bg = h$. (You should normalise h so that its leading coefficient is 1.)

   (a) $f = x^2 - 2x + 1$ and $g = x^3 + 3x^2 - 7x + 3$.

   (b) $f = x^3 + 1$ and $g = x^4 + 5x^3 + 3x^2 - 2x - 1$.

   (c) $f = x^3 - 2$ and $g = 2x^2 + 3x - 1$.

   (d) $f = x^2 + 5x + 4$ and $g = 3x^2 - 2x + 1$.

   [You may assume that F has characteristic 0, say $F = \mathbb{Q}$. It is also interesting to see what divergence we get from the general case when we set $F = \mathbb{F}_p$ for p a prime.]

2. Show that the ideal of $I = \langle x, y \rangle$ of $F[x, y]$ cannot be generated by a single element.

3. For $n \geqslant 2$ show that the ideal of $I = \langle x_1, x_2, \ldots, x_n \rangle$ of $F[x_1, x_2, \ldots, x_n]$ cannot be generated by a single element. What is the minimum number of elements required to generate I? Justify your answer.

4. Let $n, m \in \mathbb{N}$, with $n \geqslant 1$. How many n-tuples $(a_1, \ldots, a_n) \in \mathbb{N}^n$ are there such that $\sum_{i=1}^{n} a_i = m$?