Galois groups, imaginaries and definable measures.

Abstract: Algebraic closure and imaginary sorts became central objects of model theory following Shelah's finite equivalence relation theorem, showing that for a stable theory, finite Galois groups are the unique obstruction to existence of a definable type. One can also deduce definable measures are unconstrained; any type in a stable theory extends to a definable measure.

Kim and Pillay showed that the compact part of the Lascar Galois group is the unique obstruction to 3-amalgamation in simple theories. Such compact Galois groups are now well understood. By contrast, the general Lascar group remains somewhat forbidding.

I will survey these notions, and discuss the following (easy) result, part of joint work with Krupinski and Pillay:

**Proposition 0.1.** *Let $T$ be any complete theory. Assume any definable set carries a definable measure. Then Lascar types equal Kim-Pillay types. Equivalently, the general Lascar group reduces to the compact part.*

The proof relies on a lemma asserting that stable independence implies statistical independence, relative to almost any Kim-Pillay type.

Definitions: I will aim for the talk to be comprehensible by students who feel comfortable with saturated models. It will be easier if you think in advance about the main notions in the abstract. Here are the definitions.

Fix a (countable) language $L$ and a complete theory $T$. A *definable set* is a set (of n-tuples) cut out by a formula of $L$. Similarly a *type* is a maximal consistent set of formulas of $T$ (no parameters.)

A *definable measure* on a definable subset $X$ of a model $M$ is a prescription for assigning probabilities to $M$-definable subsets of $X$, for any model $M \models T$. More precisely it is an assignment $m(D)$ of real numbers in $[0,1]$ to each $M$-definable set $D$, so that $m(X) = 1$, $m(D \cup D') = m(D) \cup m(D')$ when $D, D'$ are disjoint subsets of $X$, and such that when given a formula $\phi(x,y)$, letting $D_b = \{a : M \models \phi(a,b)\}$, the function $tp(b) \mapsto m(D_b)$ is a well-defined, continuous function. This in turn means that whenever $0 < \alpha < \beta \leq 1$, there exists a definable set $W$ separating the two sets $\{b : m(D_b) < \alpha\}$ from $\{b : m(D_b) > \alpha\}$. The special case where $m$ takes values in $\{0,1\}$ is called a *definable type*.

More general Galois groups were defined by Lascar. The higher generality consists in the level of definability of the equivalence relations considered. A $\bigwedge$-*definable set* is a (countable) intersections of definable sets. A $\bigvee\bigwedge$-definable set is a countable union of $\bigwedge$-definable sets. $\bigwedge$- or $\bigvee\bigwedge$-definable sets will only be interpreted in $\aleph_1$- saturated model $M$ of $T$.

Let $E$ be an equivalence relation $E$ on a definable set $D$. Call $E$ *Shelah / Kim-Pillay /Lascar* if it is definable / $\bigwedge$-definable / $\bigvee\bigwedge$-definable, respectively, and the number of classes of $Im_E := D(M)/E$ is bounded independently of $M$;

equivalently if $M \prec M'$ are two $\aleph_1$-saturated models, then any element of $M'$ is $E$-equivalent to an element of $M$.

It follows that $Im_E$ does not depend on the choice of $M$: as long as $M, M'$ are $\aleph_1$- saturated, $Im_E(M)$ and $Im_E(M')$ can be canonically identified.

Define the Shelah / Kim-Pillay / Lascar Galois group $G_E$ associated to $Im_E$ to be the group of all permutations of $Im_E$ induced by an automorphism of some such $M$. Two tuples are said to have the same Shelah / Kim-Pillay / Lascar strong type if they are $E$-equivalent for any such $E$.