ALGEBRAIC GEOMETRY LTCC EXAM (2013)

Answer all questions.

1. [The Twisted Cubic Curve] Let k be an algebraically closed field, and let $Y \subseteq \mathbb{A}^3_k$ be the set $Y = \{(t, t^2, t^3) : t \in k\}$.

(1) Find generators for the ideal $I(Y)$ and show that Y is an affine variety in $\mathbb{A}^3$.

(2) Show that $\mathscr{O}(Y)$ is isomorphic to a polynomial ring in one variable over k. Hence show that Y is an algebraic variety of dimension 1, i.e., an algebraic curve in $\mathbb{A}^3$.

(3) Consider the projection $\pi : \mathbb{A}^3 \to \mathbb{A}^2$, described in affine coordinates by $(x, y, z) \mapsto (y, z)$. Show that the image Z of π is a closed subset of $\mathbb{A}^2$, i.e., show that Z is a plane algebraic curve.

(4) Show that the restriction of π to $Y \setminus \{(0, 0, 0)\}$ is an isomorphism onto $Z \setminus \{(0, 0)\}$. Hint: consider the corresponding affine coordinate rings!

(5) Is Y isomorphic to Z? Hint: consider the singular points!

(6) Let $\tilde{Y}$ be the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$, given in homogeneous coordinates by

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

Find the generators of the homogeneous ideal $I(\tilde{Y})$. Compare the minimal number of generators for $I(Y)$ and the minimal number of homogeneous generators for $I(\tilde{Y})$.

2. [The d-uple Embedding.] Let k be an algebraically closed field. Given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define the d-uple embedding $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$,

$$(a_0, \ldots, a_n) \mapsto (M_0(a), \ldots, M_N(a)).$$

(1) Let $\theta : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Show that $\mathfrak{a}$ is a homogeneous prime ideal, so that $V(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(2) Show that the image of ρ_d is exactly $V(\mathfrak{a})$.

(3) Show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $V(\mathfrak{a})$.

(4) Show that ρ_d is an isomorphism onto its image.

(5) Show that the twisted cubic curve in $\mathbb{P}^3$ from the previous exercise is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for a suitable choice of coordinates.

3.

(1) [Zeta function of a Pell conic] Let X be the Pell conic

$$V(x^2 - \Delta y^2 - 4) \subseteq \mathbb{A}^2_k,$$

for $\Delta \in k \setminus \{0\}$, and $k = \mathbb{F}_q$, for q a power of an odd prime. Compute the zeta function of X over k. Hint:

(a) Note that $P = (2, 0)$ is always a point on X. Show that X is a non-singular curve.

(b) In the case when Δ is a square in k, X is isomorphic to the hyperbola $V(xy - 1)$ so it is easy to count its points over finite fields.

(c) In the more interesting case when Δ is not a square in k, draw lines of varying slopes through P and compute the other intersection with X and use this 'projection from P' to count the points over k. Note that if Δ is not a square in $\mathbb{F}_q$, it will be a square in $\mathbb{F}_{q^{2r}}$ and it will not be a square in $\mathbb{F}_{q^{2r+1}}$.

(2) [Extension of scalars for zeta] Prove the *Extension of scalars Lemma* for the zeta function. Let X be a variety defined over a finite field $\mathbb{F}_q$, and let $X \times_{\mathbb{F}_q} \mathbb{F}_{q^r}$ be the same variety considered over the extension field $\mathbb{F}_{q^r}$. Then

$$Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^r}/\mathbb{F}_{q^r}, T^r) = \prod_{\xi^r=1} Z(X/\mathbb{F}_q, \xi T).$$

The product on the right is taken over the r-th roots of unity.

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