10. Exercises

1. Find a first-order axiomatisation for the class of torsion-free divisible abelian groups and prove that this theory is complete. Prove that the property 'G is torsion' cannot be axiomatised in a first-order way in the language of groups.
2. Working in the theory ACF, eliminate the quantifiers from:

$$\exists w\, \exists x\, \exists y\, \exists z(aw + by = 1 \wedge ax + bz = 0 \wedge cw + dy = 0 \wedge cx + dz = 1).$$

3. Is the theory ACF_p $\aleph_0$-categorical? Is DLO uncountably categorical?
4. Show the existence of a *countable* non-standard (non-isomorphic to $\mathbb{N}$) model of PA. Show the existence of a non-archimedean model of RCF.
5. Let θ be an $\forall\exists$-sentence, i.e.,

$$\theta \equiv \forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m\, \varphi(x_1, \dots, x_n, y_1, \dots, y_m),$$

where φ is quantifier-free. Let M_i, $i \in I$ be a chain of structures indexed by a linear order $(I, <)$ such that $M_i \models \theta$ for all $i \in I$. Prove that $\bigcup_{i \in I} M_i \models \theta$.
6. Let $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ be an algebraic automorphism of $\mathbb{C}^n$ viewed as an algebraic variety (the affine n-space $\mathbb{A}^n$). In other words, the components of σ are polynomial maps. Prove that, if $\sigma^2 = 1$, then σ has a fixed point.
7. Use model-completeness of DLO to prove that *order-completeness* (the property that every nonempty subset with an upper bound has a supremum) is not expressible in first-order logic.
8. Let F be a real closed field. An ideal $I \subseteq F[x_1, \dots, x_n]$ is *real*, if $f_1^2 + \cdots + f_m^2 \in I$ implies $f_1, \dots, f_m \in I$. Formulate and prove analogues of Theorem 16 for real prime ideals, and Proposition 7 for RCF.
9. Verify that countable models of DLO are $\aleph_0$-saturated. What is the minimum size for an $\aleph_1$-saturated model?
10. Verify the statement of Example 14.
11. It is well-known in algebra that $\mathbb{C}$ cannot be made into an ordered field. Prove that it cannot even be made into a total order by a first-order formula in the language of rings.
12. For a topological space X and an ordinal α, find the definition of the *α-th Cantor-Bendixson derivative* X^α on Wikipedia. We apply these considerations to the Stone space $X = S_n(M)$ of an $\aleph_0$-saturated model M of a complete theory T. Given a type $p \in X$, we say that its *Cantor-Bendixson rank* is α, written $CB(p) = \alpha$, if $p \in X^\alpha \setminus X^{\alpha+1}$. Prove that:

(a) $MR(p) \geq CB(p)$;
(b) if T is totally transcendental, $MR(p) = CB(p)$.