4.1

Relations and functions

Syllabus

Lookup tables, 1–1 maps, bijection and databases

4.2

A database

Studentid	Year	Code	Surname	Mark
0122	3	G503	Abbulla	56
0134	3	H641	Bosher	88
0277	2	H610	Coffey	34
0281	2	H641	Day	63
0319	1	H600	Eddy	77
0324	1	H655	Fiaz	80
0333	1	H640	Day	71

4.3

The titles at the tops of the columns (‘Studentid’ etc.) are the *attributes*.
Each attribute names a set, for example ‘Year’ names the set $\{1, 2, 3, 4\}$ of developmental years of students.

Each row below the top is a *record*.
It stores data about one individual.

4.4

Each record is a member of the product set

$$\text{Studentid} \times \text{Year} \times \text{Code} \times \text{Surname} \times \text{Mark}.$$

So the database is a subset of this 5-dimensional product.

Subsets of products are called *relations*.
So we have a *relational database*.
It is *5-ary*, i.e. of *dimension 5*.

4.5

In most relational databases,
one attribute is used to name the records.
This attribute is called the *key* (or *primary key*).

We must never have two records with the same
value of the key.
So we say a set of attributes, K, is a *candidate key* if whenever
r and s are different records,
r and s have different values for at least one attribute in K.
When K contains just one attribute,
we call that attribute a *candidate key*.

4.6

Example. In our student database,
Studentid is a candidate key.
(Almost certainly it will be the primary key.)

Mark is also a candidate key,
but this is accidental and not safe to use.

Surname is not a candidate key.
But $\{\text{Code}, \text{Surname}\}$ is a candidate key.

Any others?

4.7

Find three candidate keys:

A	B	C	D	E	F
1	3	2	5	6	7
1	4	3	6	6	1
4	2	2	1	1	1
5	6	2	4	1	7
5	4	3	7	1	8
6	3	2	3	1	3
7	6	2	9	1	3
7	2	2	8	2	8

4.8

Lookup tables

A *lookup table* is a relational database of dimension 2
where the first attribute is a candidate key.

For ‘of dimension 2’ we usually say *binary*.

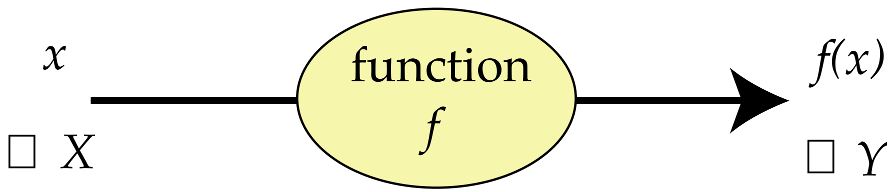
4.9

Example: A table of internet country codes

Country	Code
Afghanistan	af
Albania	al
Algeria	dz
Antarctica	aq
Austria	at
Bahrain	bh
etc.	etc.

4.10

A *function* from X to Y:



Every input x from X yields an output $f(x)$ in Y.
This output $f(x)$ depends only on the input x.

4.11

A function f from X to Y is written $f : X \to Y$.

The set X is called the *domain* of f.
The elements of X are called the *indices*.
(Mathematicians call them the *arguments*).

The outputs $f(x)$ are called the *values*.
The set of all the values of f is called the *range* of f.
We'll see many examples where the range is not the whole of Y.

4.12

Suppose we have a lookup table with attributes A, B, where every possible value of the key A is listed on the left.

Then the lookup table describes a function $f : A \to B$.

Each value a of the key is an index,
and $f(a)$ is the corresponding value in B.

In principle every function can be described this way (unless it's too large).

4.13

Example: The lookup table of a function $f : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$:

A	B
1	6
2	6
3	4

4.14

Why is the following not the lookup table of a function $g : A \to B$ where $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$?

A	B
1	6
2	6
2	5
3	4

4.15

Why is the following not the lookup table of a function $h : X \to Y$ where $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$?

X	Y
1	4
2	6

4.16

Why is the following not the lookup table of a function $k : S \to T$ where $S = \{1, 2, 3\}$ and $T = \{4, 5, 6\}$?

S	T
1	3
2	4
3	5

4.17

Reverse lookup

Suppose f is a function from X to Y.

We *query* f by giving an index x in X and asking for $f(x)$.

The opposite process, where we give a value $y \in Y$ and ask for x such that $f(x) = y$, is called *reverse lookup*.

Reverse lookup can run into two kinds of problem.

4.18

First problem: Given $y \in Y$, there may be no $x \in X$ such that $f(x) = y$.

We say that f is *onto* if every $y \in Y$ is a value of f,
i.e. for every $y \in Y$ there is $x \in X$ with $f(x) = y$.
So our first problem is that not all functions are onto.

Example: The function

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$$

is not onto, because every square is $\geqslant 0$,
so we can't do a reverse lookup on -1.

4.19

Second problem: Given $y \in Y$, there can be two or more $x \in X$ such that $f(x) = y$.
In this case a reverse lookup could return a list of the relevant x,
since there is no one right answer.

We say that f is *one-to-one* (or *1–1*) if $f(x_1) = f(x_2)$ always implies $x_1 = x_2$.
So our second problem is that not all functions are one-to-one.

4.20

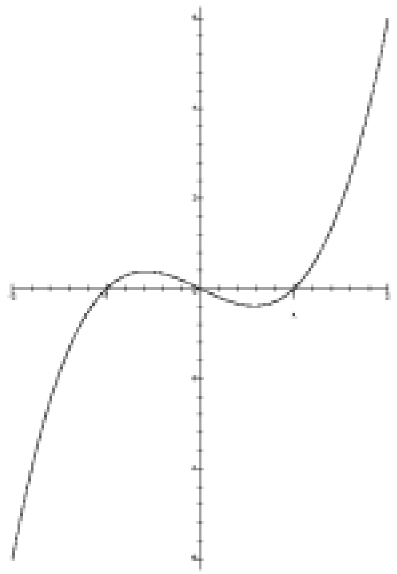
Example: The function

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - x$$

is onto, but it isn't 1–1 since

$$f(0) = 0 = f(1).$$

4.21



4.22

When a function $f : X \to Y$ is one-to-one and onto, we say it is a *bijection*.

For a bijection $f : X \to Y$, reverse lookup always works. In fact reverse lookup defines a function $g : Y \to X$ so that

$$f(x) = y \text{ if and only if } g(y) = x.$$

This function g is called the *inverse* of f, written f^{-1}.

4.23

If $f : X \to Y$ has a lookup table and an inverse f^{-1}, then the lookup table of f^{-1} is the table of f but with left and right reversed.

f :

X	Y
1	8
2	7
3	2
4	9

f^{-1} :

Y	X
8	1
7	2
2	3
9	4

4.24

When $f : \mathbb{R} \to \mathbb{R}$, we can sometimes show that f is a bijection by calculating its inverse:

- Write y for $f(x)$ in the equation for $f(x)$.
- Rearrange the equation so that it reads $x = \ldots$.

4.25

Example: $f(x) = 4x - 8$.

Write

$$y = 4x - 8.$$

Rearrange and divide by 4:

$$\begin{aligned} 4x &= y + 8. \\ x &= \tfrac{y}{4} + 2. \end{aligned}$$

So the inverse of f is

$$g : \mathbb{R} \to \mathbb{R}, \quad g(y) = \frac{y}{4} + 2.$$

4.26

Example

Let f be the following function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4\}$:

$$f(1) = 3, \; f(2) = 4, \; f(3) = 4, \; f(4) = 1.$$

Then f is not one-to-one, because $f(2) = f(3)$.
Also it is not onto, because there is no x with $f(x) = 2$.

4.27

Example

Let f be the same function as before, namely

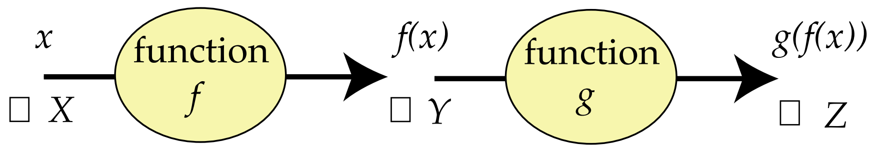
$$f(1) = 3, \; f(2) = 4, \; f(3) = 4, \; f(4) = 1,$$

but regarded as a function from $\{1, 2, 3, 4\}$ to $\{1, 3, 4\}$.
Then f is onto, but it is still not one-to-one.

4.28

Composing functions

Given a function $f : X \to Y$ and a function $g : Y \to Z$,
we can set them end-to-end and get a function $h : X \to Z$:



So for all $x \in X$, $h(x) = g(f(x))$.
We write h as $g \circ f$, and we call it the *composite* of f and g.

4.29

Example

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x^2,\ g(x) = \sqrt{x^2 + 2}.$$

Then we form $g \circ f$ by substituting $f(x)$ for x in g:

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= \sqrt{f(x)^2 + 2} \\ &= \sqrt{(x^2)^2 + 2} \\ &= \sqrt{x^4 + 2}.\end{aligned}$$

4.30

If $f : X \to Y$ and $g : Y \to Z$ have lookup tables, then we calculate the lookup table of $g \circ f$ by

- taking each x in X,
- looking up $f(x)$ in the table for f,
- then looking up $g(f(x))$ in the table for g.

4.31

Example

f :

X	Y
1	8
2	7
3	2
4	8
5	2

g :

Y	Z
2	3
3	4
5	6
7	8
8	8

$g \circ f$:

X	Z
1	8
2	8
3	3
4	8
5	3