3.1

Sets

Syllabus

Sets: subsets, unions, intersections

3.2

Naming sets

To describe a small set,
we list its members between curly brackets $\{, \}$:

$\{2, 4, 6, 8\}$
$\{$ England, France, Iran, Singapore, New Zealand $\}$
$\{$ David Beckham $\}$
$\{\}$ (the empty set, also written $\emptyset$)

We write $a \in X$ to express that a is a member of the set X.
For example $4 \in \{2, 4, 6, 8\}$.
$a \notin X$ means a is not a member of X.

3.3

Two sets are equal if they have the same members.
The order and the number of times the members are mentioned are irrelevant.

For example

$$\begin{aligned} \{2,3,2,4\} &= \{4,4,2,3,4\}. \\ \{\text{England, France, France}\} &= \{\text{France, England}\}. \\ \{2,3,2,4\} &\neq \{3,4,5\}. \end{aligned}$$

3.4

There is another way of naming sets,
that we can use when the sets are too large to list.

$$\{x : x \text{ is an integer divisible by } 3\}$$

means the set of all integers that are divisible by 3.
The x can be replaced all through by another letter. So

$$\{n : n \text{ is an integer divisible by } 3\}$$

is the same set as above.

3.5

The following question in the 2004 Exam confused some people:

> Let A be the set $\{2, 3, 4\}$ and let C be the set of all numbers that are twice some number in A.
> Write out the set C.

Answer this question.

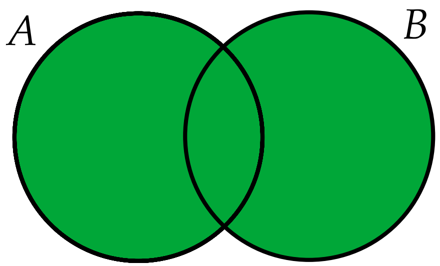
3.6

Combining sets

Suppose A and B are sets. We write $A \cup B$ for the set

$$\{x : x \in A \text{ or } x \in B\}.$$

This set is called the *union* of A and B.

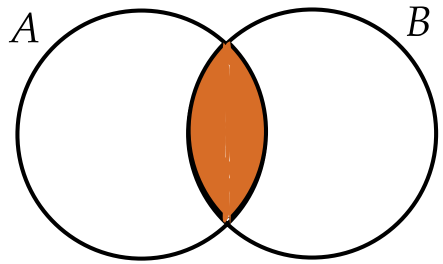


3.7

We write $A \cap B$ for the set

$$\{x : x \in A \text{ and } x \in B\}.$$

This set is called the *intersection* of A and B.



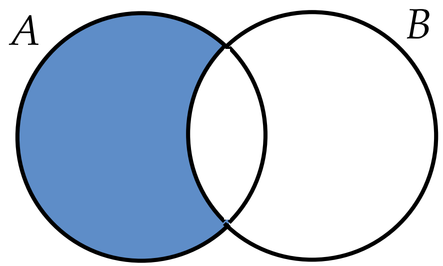
Mnemonic: $\cup$ goes with $\vee$ ('or') and $\cap$ goes with $\wedge$ ('and').

3.8

We write $A \setminus B$ for the set

$$\{x : x \in A \text{ and } x \notin B\}.$$

This set is called the *complement* of B in A.
When A is fixed, we write just B' for $A \setminus B$.



3.9

Examples

$$
\begin{aligned}
\{1,2,3\} \cup \{2,4,6\} &= \{1,2,3,4,6\} \\
\{1,2,3\} \cap \{2,4,6\} &= \{2\} \\
\{6,7,9\} \cup \emptyset &= \{6,7,9\} \\
\{6,7,9\} \cap \emptyset &= \emptyset \\
\{1,2,3\} \setminus \{2,4,6\} &= \{1,3\} \\
\{1,2,3\} \setminus \emptyset &= \{1,2,3\}
\end{aligned}
$$

3.10

More examples to calculate

$$
\begin{aligned}
\{0,2,3,5,7,10,12\} \cup \{0,1,2,4,6,7,10,11\} &= \\
\{0,2,3,5,7,10,12\} \cap \{0,1,2,4,6,7,10,11\} &= \\
\{0,2,3,5,7,10,12\} \setminus \{0,1,2,4,6,7,10,11\} &= \\
\{5,8,8,9\} \cup \emptyset &= \\
\{6,7,8,9,10\} \cap \emptyset &= \\
\emptyset \setminus \{1,2,3\} &=
\end{aligned}
$$

3.11

Some basic facts about combining sets

Commutative laws:

$$
\begin{aligned}
A \cup B &= B \cup A \\
A \cap B &= B \cap A
\end{aligned}
$$

Idempotence laws:

$$A \cup A = A = A \cap A.$$

3.12

Associative laws:

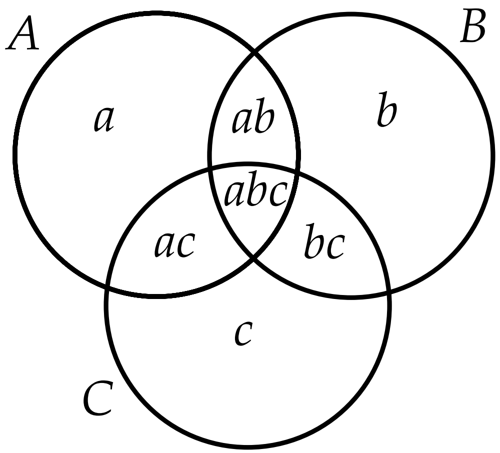
$$
\begin{aligned}
(A \cup B) \cup C &= A \cup (B \cup C) \\
(A \cap B) \cap C &= A \cap (B \cap C)
\end{aligned}
$$

Because of these laws we can leave out some brackets and write for example

$$A \cup B \cup C \cup D; \quad U \cap V \cap W \cap X \cap Y.$$

3.13

Three sets together



3.14

Distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

We check this from the diagram in 3.13:

$B \cup C$ consists of b, c, ab, ac, bc, abc.
So $A \cap (B \cup C)$ consists of ab, ac, abc.

$A \cap B$ consists of ab, abc, and $A \cap C$ consists of ac, abc.
So $(A \cap B) \cup (A \cap C)$ consists of ab, ac, abc,
the same as $A \cap (B \cup C)$.

3.15

We also have the distributive law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

This is the same as the other distributive law,
but with $\cup$ and $\cap$ reversed.

3.16

Cardinality

We write $|A|$ for the number of members of the set A.
$|A|$ is called the *cardinality* of A.

Examples:

$$\begin{aligned} |\{5, 9, 9, 10\}| &= 3, \\ |\{81, 1, 5, 77, 49\}| &= 5, \\ |\emptyset| = 0. & \end{aligned}$$

3.17

We say that two sets A and B are *disjoint* if $A \cap B = \emptyset$.

If A and B are disjoint then

$$|A \cup B| = |A| + |B|.$$

For example in the diagram 3.13,

$$|A| = |a| + |ab| + |ac| + |abc|.$$

3.18

The *principle of inclusion-exclusion* says:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

To check this:

$$\begin{aligned}
& |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \\
= & (|a| + |ab| + |ac| + |abc|) + (|b| + |ab| + |bc| + |abc|) \\
& +(|c| + |ac| + |bc| + |abc|) - (|ab| + |abc|) \\
& -(|bc| + |abc|) - (|ac| + |abc|) + |abc| \\
= & |a| + |b| + |c| + (2-1)|ab| + (2-1)|bc| \\
& +(2-1)|ac| + (3-3+1)|abc| \\
= & |A \cup B \cup C|.
\end{aligned}$$

3.19

Example (based on Exam 2003) In a survey of 140 people it was found that 115 people regularly use a telephone; 39 people regularly write letters; 64 people regularly use e-mail; 28 people regularly use a telephone and write letters; 53 people regularly use a telephone and e-mail; 10 people regularly use e-mail and regularly write letters; and 3 people regularly write letters and use a telephone and e-mail. How many people in the survey do not regularly use telephone or letters or e-mail? How many people do not regularly use either telephone or e-mail?

3.20

Solution

Put

$T =$ people who regularly use telephone

$L =$ people who regularly write letters,

$E =$ people who regularly use e-mail.

Then $|T| = 115$, $|L| = 39$, $|E| = 64$, $|T \cap L| = 28$, $|T \cap E| = 53$, $|E \cap L| = 10$, $|T \cap L \cap E| = 3$.

3.21

So by the principle of inclusion-exclusion,

$$\begin{aligned} |T \cup L \cup E| &= 115 + 39 + 64 - 28 - 53 - 10 + 3 \\ &= 130. \end{aligned}$$

So the number of people who don't regularly use any of telephone, letters or e-mail is $140 - 130 = 10$.

The number of people who don't regularly use either telephone or e-mail is 140 minus the number of people who do regularly use at least one of these.
Calculate this.

3.22

Example Suppose

$$|A| = 110,\ |B| = 91,\ |C| = 84,$$
$$|A \cap B| = 71,\ |A \cap C| = 57,\ |B \cap C| = 53,$$
$$|A \cap B \cap C| = 51.$$

Find $|A \cup B \cup C|$.

3.23

Example: Suppose

- 10 people play the guitar,
- 16 people play the piano,
- 15 people play the drums,
- 7 people play the guitar and the piano,
- 8 people play the guitar and the drums,
- 6 people play the piano and the drums,
- 25 people play at least one of the guitar, the piano and the drums.

How many people play all three instruments?
How many people play at least one of piano and drums?

3.24

Subsets

Suppose A and B are sets.
We say that A is a *subset* of B if every member of A is also a member of B.

Example: The subsets of $\{0, 1, 2, 3\}$ are

$$\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}, \{2,3\},$$
$$\{0,1,2\}, \{0,1,3\}, \{0,2,3\}, \{1,2,3\}, \{0,1,2,3\}.$$

The number of subsets of an n-element set is

$$2^n.$$

3.25

Products

An *ordered pair* is a list of length 2.
We write (x, y) for the ordered pair which lists x first and y second.
We can have (a, a) (the same element listed twice).

We have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

We write $A \times B$ for the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$.

3.26

Example

Suppose A is $\{1, 2, 3\}$ and B is $\{4, 5\}$.
Then $A \times B$ is

$$\{(1, 4), (2, 4)(3, 4), (1, 5), (2, 5), (3, 5)\}.$$

In this example $|A| = 3$ and $|B| = 2$, while $|A \times B| = 6$.
This suggests the law

$$|A \times B| = |A| \times |B|.$$

The law is true and we prove it as follows
(for finite sets A and B).

3.27

Suppose A has m members, $a_1, \ldots, a_m$,
and B has n members, $b_1, \ldots, b_n$.
We can arrange the members of $A \times B$ in a rectangle:

$$\begin{array}{cccc}
(a_1, b_1) & (a_1, b_2) & \ldots & (a_1, b_n) \\
(a_2, b_1) & (a_2, b_2) & \ldots & (a_2, b_n) \\
\vdots & \vdots & & \vdots \\
(a_m, b_1) & (a_m, b_2) & \ldots & (a_m, b_n)
\end{array}$$

m rows and n columns gives mn elements.

3.28

The law

$$|A \times B| = |A| \times |B|$$

is very useful when we have to list
all the members of $A \times B$.
It gives us a check on whether we have included
all the members.
In computer programs it gives a bound on
how many members we need to list.

3.29

We write

$$A^2$$

for the product set $A \times A$. If $|A| = m$ then

$$|A^2| = |A \times A| = |A| \times |A| = m^2.$$

Example: We write $\mathbb{R}$ for the set of real numbers.
Then $\mathbb{R}^2$ is the xy plane.
Each point (a, b) has x-coordinate a and y-coordinate b.

3.30

Higher-dimensional products

An *ordered triple* is written (a, b, c).
If A, B and C are sets, then we write

$$A \times B \times C$$

for the set of ordered triples (a, b, c) such that
$a \in A$, $b \in B$ and $c \in C$.

We write A^3 for $A \times A \times A$.
For example $\mathbb{R}^3$ is 3-dimensional space.

Similarly with higher dimensions.