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A SUFFICIENT CONDITION FOR THE SUBORDINATION PRINCIPLE IN ERGODIC OPTIMIZATION

I.D. MORRIS

Abstract

Let $T\colon X \to X$ be a continuous surjection of a topological space, and let $f\colon X \to \mathbb{R}$ be upper semi-continuous. We wish to identify those T-invariant measures μ which maximize $\int f d\mu$. We call such measures f*-maximizing*, and denote the maximum by $\beta(f)$. The study of such measures and their properties has recently been dubbed *ergodic optimization*.

A first step to understanding the structure of a function's maximizing measures is to establish the following *subordination principle* defined by T. Bousch: if μ and ν are T-invariant measures such that $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$ and μ is f-maximizing, then ν is also f-maximizing. Previous authors have approached this result by constructing a continuous function $g\colon X \to \mathbb{R}$ such that $f - \beta(f) \leq g \circ T - g$. We provide a sufficient condition for the subordination principle which has advantages when the space X is noncompact.

1. *Introduction*

Let $T\colon X \to X$ be a continuous surjection of a topological space, and let $\mathcal{M}_T$ denote the set of all T-invariant Borel probability measures on X. Let $f\colon X \to \mathbb{R}$ be upper semi-continuous. We define the *maximum integral* of f to be $\beta(f) := \sup_{\mu \in \mathcal{M}_T} \int f d\mu$, and say that an invariant measure μ is *maximizing* if $\int f d\mu = \beta(f)$. We wish to study the set of f-maximizing invariant measures, which we denote by $\mathcal{M}_{max}(f)$.

We say that the maximizing measures of f satisfy the *subordination principle* [**3**] if the following property holds: if $\nu \in \mathcal{M}_T$, $\mu \in \mathcal{M}_{max}(f)$ and $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$, then $\nu \in \mathcal{M}_{max}(f)$. We say that a closed set $K \subseteq X$ is a *maximizing set* for f if it has the property that $\mu \in \mathcal{M}_T$ is f-maximizing if and only if $\operatorname{supp}\mu \subseteq K$. It is not difficult to see that the existence of an f-maximizing set implies the subordination principle.

In the existing literature on maximizing measures, the subordination principle has exclusively appeared as a corollary of the following type of result: if $f\colon X \to \mathbb{R}$ satifies some form of regularity (for example, f is Hölder continuous with respect to some natural metric) then there is a continuous g such that $f - \beta(f) \leq g \circ T - g$; theorems of this type are widespread in the literature, see e.g. [**3**, **5**, **7**, **8**]. From such a result it then follows that $f - \beta(f) = g \circ T - g + r$ for some continuous (or upper semi-continuous) function $r \leq 0$, whereupon we see that the closed set $K := r^{-1}\{0\}$ is f-maximizing. Hence, if $\operatorname{supp}\nu \subseteq \operatorname{supp}\mu$ and μ is maximizing, $\operatorname{supp}\nu \subseteq K$ and ν is maximizing. In hyperbolic situations, constructions of this type succeed for a large class of 'regular' functions [**3**], but it is known that the subordination principle fails in a residual subset of continous functions [**4**].

(In fact, in the above situation we use only the facts that $\int (g \circ T - g)\, d\mu = 0$

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for all $\mu \in \mathcal{M}_T$ and $g \circ T - g$ is continuous, and so do not really use the fact that $g \circ T - g$ is a coboundary. We discuss some implications of this fact in section 3.)

A natural question is to examine the extent to which the subordination principle necessitates the existence of a continuous coboundary $g \circ T - g \geq f - \beta(f)$. In this note, we provide a new sufficient condition for the subordination principle which, in the case where the space X is compact, is strictly weaker than the existence of such a coboundary.

For $n > 0$ we write $S_n f := \sum_{j=0}^{n-1} f \circ T^j$. Our principal result is the following:

THEOREM 1. *Let $f \colon X \to \mathbb{R}$ be upper semi-continuous with $\beta(f) \neq \pm\infty$, and suppose that $\sup_{n \geq 1} \sup_{x \in X} S_n[f - \beta(f)](x) < \infty$. Then $\mathcal{M}_{max}(f)$ satisfies the subordination principle.*

In section 2 of this note we prove Theorem 1, and deduce the existence of a closed f-maximizing set. We also provide an application to the ergodic optimization of countable-state subshifts of finite type as studied in [**7**]. In section 3, we prove two supplementary propositions: firstly, that the hypotheses of the theorem do not imply the existence of $g \in C(X)$ such that $f - \beta(f) \leq g \circ T - g$; and secondly, that the converse of the theorem is false.

An overview of existing results in ergodic optimization may be found in [**6**].

2. *Proof and applications of Theorem 1*

Proof. We will prove the following statement: if $\nu, \mu \in \mathcal{M}_T$ satisfy $\operatorname{supp} \nu \subseteq \operatorname{supp} \mu$ and $\int f d\nu < \beta(f)$, then $\int f d\mu < \beta(f)$. We assume without loss of generality that $\beta(f) = 0$. Let $B \geq \sup_{n \geq 1} \sup_{x \in X} S_n f(x)$ with $B > 0$.

Let $\nu, \mu \in \mathcal{M}_T$ with $\operatorname{supp} \nu \subseteq \operatorname{supp} \mu$ and $\int f d\nu < 0$. By the Birkhoff ergodic theorem applied to ν, there exists at least one point $x_0 \in \operatorname{supp} \nu$ such that $\lim_{n \to \infty} \frac{1}{n} S_n f(x_0) = \int f d\nu < 0$; take $N > 0$ such that $S_N f(x_0) \leq -3B$. Since f is upper semi-continuous, there exists an open set $U \ni x_0$ such that $S_N f(x) < -2B$ for all $x \in U$.

By the ergodic decomposition theorem for measurable spaces [**1**, Theorem 2.2.8], there is a probability space $(\Omega, \mathcal{F}, m)$ and a collection of T-invariant ergodic Borel probability measures $\{\mu_\omega \colon \omega \in \Omega\}$ on X such that for every Borel set $Z \subseteq X$, the map $\omega \mapsto \mu_\omega(Z)$ is $\mathcal{F}$-measurable and

$$\mu(Z) = \int_\Omega \mu_\omega(Z) dm(\omega).$$

Suppose $\mu_\omega(U) > 0$. For $x \in U$, define the sequence of return times $r_n(x)$ in the following way: let $r_1(x) = 0$, and for $n \geq 1$ define $r_n(x) = \inf\{k > r_{n-1}(x) \colon T^k x \in U\}$. Applying the ergodic theorem and Kac's lemma to the ergodic measure μ_ω, we can find $x_\omega \in U$ such that as $n \to \infty$, $r_n(x_\omega) \sim \frac{n}{\mu_\omega(U)}$ and $\frac{1}{n} S_n f(x_\omega) \to \int f d\mu_\omega$.

Let $n > 0$. The number of integers k satisfying $0 \leq k \leq r_n(x_\omega)$ such that $T^k x_\omega \in U$ is equal to precisely n. We may therefore choose an increasing sequence of $M_n = \lfloor \frac{n}{N} \rfloor$ integers $(n_i)_{i=1}^{M_n}$ with the following properties: $n_1 = 0$, $n_{M_n} = r_n(x_\omega)$, and for each i, $T^{n_i} x_\omega \in U$ and $n_{i+1} \geq n_i + N$. Using our hypothesis that $S_n f \leq B$

together with the definition of U we obtain

$$\begin{aligned} S_{r_n(x_\omega)}f(x_\omega) &= \sum_{i=1}^{M_n-1}\left(\sum_{j=n_i}^{n_i+N-1} f(T^j x_\omega) + \sum_{j=n_i+N}^{n_{i+1}-1} f(T^j x_\omega)\right) \\ &\leq \sum_{i=1}^{M_n-1}(-2B+B) = -B(M_n-1) \end{aligned}$$

for each $n>0$. Hence

$$\begin{aligned} \int f d\mu_\omega &= \lim_{n\to\infty}\frac{1}{r_n(x_\omega)}S_{r_n(x_\omega)}f(x_\omega) \\ &\leq \lim_{n\to\infty} -\frac{B(M_n-1)}{r_n(x_\omega)} \\ &= \lim_{n\to\infty} -\frac{Bn}{Nr_n(x_\omega)} = -\frac{B}{N}\mu_\omega(U), \end{aligned}$$

whenever $\mu_\omega(U)>0$. It follows from $x_0\in U\cap\operatorname{supp}\mu$ and the fact that U is open that $\mu(U)>0$, and therefore,

$$\begin{aligned} \int f d\mu &= \int_\Omega \int f d\mu_\omega dm(\omega) \\ &= \int_{\{\omega\colon \mu_\omega(U)>0\}}\int f d\mu_\omega dm(\omega) + \int_{\{\omega\colon \mu_\omega(U)=0\}}\int f d\mu_\omega dm(\omega) \\ &\leq -\int_{\{\omega\colon \mu_\omega(U)>0\}}\frac{B}{N}\mu_\omega(U)dm(\omega) + \int_{\{\omega\colon \mu_\omega(U)=0\}}\beta(f)dm(\omega) \\ &= -\frac{B}{N}\mu(U)+0<0. \end{aligned}$$

This proves the theorem. □

Our result makes it easy to check the subordination principle for a large class of observables in any dynamical system where the supply of periodic orbits is sufficiently rich. Moreover, our method does not require functional-analytic constructions which may become difficult when the space X is noncompact [**7**]. We provide an example in the context of countable-state subshifts of finite type.

Let $A\colon \mathbb{N}\times\mathbb{N}\to\{0,1\}$ be an infinite matrix, and denote its $(i,j)^{th}$ entry of A by $[A]_{i,j}$. Define a set $\Sigma_A\subseteq\mathbb{N}\times\mathbb{N}$ by

$$\Sigma_A=\{x=(x_i)_{i\geq 1}\colon [A]_{x_i,x_{i+1}}=1 \text{ for all } i\geq 1\},$$

and let $\sigma\colon\Sigma_A\to\Sigma_A$ be the *shift map* defined by $\sigma(x_i)_{i\geq1}=(x_{i+1})_{i\geq1}$. We equip Σ_A with the standard metric

$$d\left((x_i)_{i\geq1},(y_i)_{i\geq1}\right)=2^{-\inf\{i\colon x_i\neq y_i\}},$$

which makes $\sigma\colon\Sigma_A\to\Sigma_A$ continuous. Finally, for $f\colon\Sigma_A\to\mathbb{R}$ and $j\geq1$ we define

$$\operatorname{var}_j f = \sup_{\{x,y\in\Sigma_A\colon x_i=y_i\ \forall\ 1\leq i\leq j\}}|f(x)-f(y)|.$$

We have the following result:

Corollary 1. *Suppose that the shift map $\sigma\colon\Sigma_A\to\Sigma_A$ is surjective, and that the following property holds: there is an integer $N>0$ such that for every $i,j>0$*

we may find $0 < m(i,j) \leq N$ *such that* $[A^{m(i,j)}]_{i,j} \geq 1$. *Suppose that* $f\colon \Sigma_A \to \mathbb{R}$ *has* $\sum_{j=1}^{\infty} \mathrm{var}_j f < \infty$ *and* $|f|_\infty < \infty$. *Then* f *satisfies the subordination principle.*

Proof. The maximum integral $\beta(f)$ exists and is finite since $|\beta(f)| \leq |f|_\infty < \infty$; we assume without loss of generality that $\beta(f) = 0$. The condition on A implies that for every $x \in \Sigma_A$ and $n > 0$, we may find a non-negative integer $k < N$ and periodic point $p \in \mathrm{Fix}_{n+k}$ such that x and p agree in their first n symbols. Let $x \in \Sigma_A$ and $n > 0$, and choose such a periodic orbit p. We have

$$\begin{aligned} S_n f(x) &= S_{n+k} f(x) - S_k f(T^n x) \\ &\leq S_{n+k} f(p) + \sum_{j=1}^{n} \mathrm{var}_j f + 2k|f|_\infty - k \inf f \\ &\leq \sum_{j=1}^{\infty} \mathrm{var}_j f + 3N|f|_\infty, \end{aligned}$$

where we have used $S_{n+k} f(p) \leq 0$ since $\beta(f) = 0$. Applying Theorem 1 completes the proof. $\square$

We remark that while our assumptions on the matrix A are weaker than those required in [**7**], our assumption that $\inf f > -\infty$ is not present in that article.

It is not difficult to see that the above argument may be applied whenever f satisfies the Walters condition [**3**, **9**] and the dynamical system $T\colon X \to X$ satisfies the following property: for any $\varepsilon > 0$, there is $N_\varepsilon > 0$ such that for every $n > 0$, $\bigcup_{i=n}^{n+N_\varepsilon} \mathrm{Fix}_i$ is an (n, ε)-spanning set [**10**].

It should also be noted that the property $f - \beta(f) \leq g \circ T - g$ has implications additional to the subordination principle. One particular such implication is the existence of an f-maximizing set: if $f - \beta(f) \leq g \circ T - g$ where g is continuous, then the closed set $\{x \in X\colon f(x) = g(Tx) - g(x)\}$ is f-maximizing. *A priori*, it is not obvious whether or not the hypotheses of Theorem 1 imply the existence of an f-maximizing set. In fact we have the following:

PROPOSITION 1. *Let* $f\colon X \to \mathbb{R}$ *be upper semi-continous, where* X *is a separable metric space. Then an* f*-maximizing set exists if and only if* f *satisfies the subordination principle.*

Proof. The forward implication is simple: if an f-maximizing set K exists, then $\mu \in \mathcal{M}_{max}(f)$ and $\mathrm{supp}\,\nu \subseteq \mathrm{supp}\,\mu$ imply that $\mathrm{supp}\,\nu \subseteq K$ so that ν is f-maximizing.

Suppose that f satisfies the subordination principle. Since f is upper semi-continuous and X is separable, $\mathcal{M}_{max}(f)$ is closed and separable in the weak-* topology [**2**]. Let $(\mu_n)_{n\geq1}$ be a sequence of measures (not necessarily distinct) which is dense in $\mathcal{M}_{max}(f)$, and take $\mu := \sum_{n=1}^{\infty} 2^{-n}\mu_n$. Define $K = \mathrm{supp}\,\mu$; clearly, μ is f-maximizing and K is T-invariant. We claim that $\nu \in \mathcal{M}_{max}(f)$ if and only if $\mathrm{supp}\,\nu \subseteq K$.

If $\nu \in \mathcal{M}_T$ satisfies $\mathrm{supp}\,\nu \subseteq K$, then it is maximizing by the subordination principle. Conversely, suppose that ν is maximizing. Since every μ_n clearly has $\mathrm{supp}\,\mu_n \subseteq \mathrm{supp}\,\mu$, we have $\mu_n(K) = 1$ for all $n \geq 1$; but ν is a weak-* limit point of $(\mu_n)_{n\geq1}$ and K is closed, so $\nu(K) = 1$. This completes the proof. $\square$

3. *Relationship with other sufficient conditions*

In this section we restrict our attention to compact spaces X.

By considering the remarks made in the introduction, we can discern three distinct sufficient conditions for an upper semi-continuous function $f\colon X \to \mathbb{R}$ to satisfy the subordination principle: firstly, the boundedness hypothesis of Theorem 1; secondly, that $f - \beta(f) \leq h$ for some continuous coboundary h; and thirdly, more generally, that $f - \beta(f) \leq w$ for some continuous w such that $\int w d\mu = 0$ for all $\mu \in \mathcal{M}_T$. Following [**4**], we term such a function w a *weak coboundary*. This motivates the following definitions:

Condition (B): $\sup_{n\geq 1} \sup_{x\in X} S_n[f-\beta(f)](x) < \infty$
Condition (C): $f - \beta(f) \leq g\circ T - g$ for some $g \in C(X)$
Condition (W): $f - \beta(f) \leq w$ for some weak coboundary $w \in C(X)$.

Clearly, if f satisfies condition (C), then it satisfies both (B) and (W). In this section, we establish that in general, (B) does not imply (C), and (W) does not imply (B). The former result shows that Theorem 1 has greater reach than previous approaches to the subordination principle; the latter shows that the converse of Theorem 1 is in certain cases false.

Remark 1. If f satisfies condition (B), then by defining $\psi_f(x) := \sup_{n\geq 1} S_n[f-\beta(f)](x)$ one may obtain $f - \beta(f) \leq \psi_f \circ T - \psi_f$, where ψ_f is bounded; however, since ψ is not in general continuous (see below) the subordination principle does not directly follow.

We will again construct our examples in the context of shift maps. For a natural number $k \geq 2$, let $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbb{N}}$, and let $\sigma\colon \Sigma_k \to \Sigma_k$ be the shift map $(x_n)_{n\geq 1} \mapsto (x_{n+1})_{n\geq 1}$. We again use the metric

$$d\left((x_i)_{i\geq 1}, (y_i)_{i\geq 1}\right) = 2^{-\inf\{i\colon x_i\neq y_i\}},$$

which makes Σ_k compact. Both of the succeeding propositions hold true for arbitrary $k \geq 2$, but are most simply stated in their respective forms.

Proposition 2. *There exists $f \in C(\Sigma_3)$ which satisfies condition (B) but not condition (C).*

Proof. For $k \geq 1$ let C_k be the set of all sequences $(x_i)_{i\geq 1} \in \Sigma_3$ such that $x_i = 1$ when $1 \leq i \leq 2k^2$, $x_i = 2$ when $2k^2 + 1 \leq i \leq 2k^2 + k$, and $x_i = 3$ when $i = 2k^2 + k + 1$. Define

$$f = \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{\ell=0}^{k^2-1} \mathbb{1}_{\sigma^\ell C_k}\right) - \mathbb{1}_{\{(x_i)_{i\geq 1}\colon x_1=3\}},$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A. The definition above implies that if $k_1, k_2 > 0$ and $0 \leq \ell_1 < k_1^2$, $0 \leq \ell_2 < k_2^2$ then the sets $\sigma^{\ell_1} C_{k_1}$ and $\sigma^{\ell_2} C_{k_2}$ intersect only if $k_1 = k_2$ and $\ell_1 = \ell_2$. Since each $\sigma^\ell C_k$ is both closed and open, this ensures that the function $\frac{1}{k^2}\sum_{\ell=0}^{k^2-1}\mathbb{1}_{\sigma^\ell C_k}$ is continuous with norm $\frac{1}{k^2}$ for each $k>0$. It follows that f is continuous.

We claim that $\beta(f)=0$. If x does not lie in $\bigcup_{k\geq 1}\bigcup_{\ell=0}^{k^2-1}\sigma^\ell C_k$ it is clear that $f(x)\leq 0$. If $x\in\sigma^\ell C_k$ where $\ell<k^2$, then by inspection one may calculate $\sum_{j=0}^{n-1}f(\sigma^j x)\leq 1$ for all $0\leq n\leq 2k^2+k-\ell$ and $\sum_{j=0}^{2k^2+k-\ell}f(\sigma^j x)=0$. Thus $S_nf(x)$ may not exceed 1 for any $n\geq 1$ and $x\in X$. Consequently $\beta(f)\leq 0$ by the Ergodic Theorem, and since f is zero at the fixed point $z:=(1)_{n\geq 1}$ we have $\beta(f)=0$. Therefore, f satisfies $S_nf(x)\leq n\beta(f)+1$ for every $x\in X$ and $n\geq 1$, which implies condition (B).

Suppose that f satisfies condition (C), that is that $f\leq g\circ\sigma-g$ for some $g\in C(\Sigma_3)$. For $k\geq 1$ let $x^k\in C_k$, noting that this implies $S_{k^2}f(x^k)=1$. We have

$$g(T^{k^2}x^k)\geq g(x^k)+S_{k^2}f(x^k)=g(x^k)+1,$$

but since the sequences $\sigma^{k^2}x^k$ and x^k both converge to z as $k\to\infty$, we obtain $g(z)\geq g(z)+1$, an absurdity. □

Let $W(\Sigma_2)$ denote the set of weak coboundaries, that is, continuous functions $w\colon\Sigma_2\to\mathbb{R}$ such that $\int w\,d\mu=0$ for every $\mu\in\mathcal{M}_\sigma$. It is known [**4**] that $W(\Sigma_2)$ equals the uniform closure of the set of coboundaries; in particular $W(\Sigma_2)$ is complete with respect to the uniform metric. Clearly every $w\in W(\Sigma_2)$ satisfies condition (W).

Proposition 3. *The set of all $f\in W(\Sigma_2)$ which do not satisfy condition (B) is residual in $W(\Sigma_2)$.*

Proof. We first show that such an f exists, by adapting a construction due to M. Zinsmeister which is described in [**4**]. Define $\phi:=\mathbb{1}_{\{x_1=1\}}-\mathbb{1}_{\{x_1=2\}}$, and let $(r_n)_{n\geq 0}$ be any sequence of real numbers with the following properties:

(i) $r_0=0$
(ii) $r_{n+1}\leq r_n$ for all $n\geq 1$
(iii) $\sum_{n=0}^{\infty}|r_n-r_{n+1}|<\infty$
(iv) $\sup_{n\geq 1}nr_n=+\infty$.

For example, we could take $r_n=1/\sqrt{n}$ for $n\geq 1$ and $r_0=0$. Define

$$f:=\sum_{n=0}^{\infty}(r_n-r_{n+1})\,\phi\circ\sigma^n,$$

which is continuous by (iii). It is a limit of coboundaries since, using (i),

$$\begin{aligned}\sum_{k=0}^{n-1}r_{k+1}\phi\circ\sigma^{k+1}-\sum_{k=0}^{n-1}r_{k+1}\phi\circ\sigma^k &= \sum_{k=1}^{n}(r_k-r_{k+1})\,\phi\circ\sigma^k\\ &\quad+r_0\phi-r_1\phi+r_{n+1}\phi\circ\sigma^n\\ &=f-\sum_{k=n+1}^{\infty}(r_k-r_{k+1})\,\phi\circ\sigma^k+r_{n+1}\phi\circ\sigma^n,\end{aligned}$$

which converges to f by (iii). This implies $f\in W(\Sigma_2)$.

For each $m>0$, let $p^m=(p_i^m)_{i\geq 1}$, where $p_i^m=2$ for $1\leq i\leq m$ and $p_i^m=1$ for all $i > m$. Using (ii), we calculate

$$\begin{aligned} S_{2m}h(p^m) &= \sum_{n=0}^{\infty}\sum_{j=0}^{2m-1}(r_n - r_{n+1})\phi(\sigma^{n+j}p_m) \\ &= \sum_{n=0}^{m-1} 2na_n + \sum_{n=m}^{\infty} 2ma_n \\ &= 2\sum_{n=0}^{m-1}(nr_n - nr_{n+1}) + 2mr_m \\ &\geq 2mr_m; \end{aligned}$$

but this is unbounded by (iv), so $\sup_{k\geq 1}\sup_{x\in\Sigma_2} S_k h(x) = \infty$ and f does not satisfy (B).

The remainder of the proof follows easily. For $n \geq 1$, define $K_n := \{w \in W(\Sigma_2)\colon \sup_{k\geq 1}\sup_{x\in\Sigma_2}(S_k w)(x) \leq n\}$. Clearly each K_n is closed, and $f \in W(\Sigma_2)$ satisfies (B) if and only if it lies in some K_n. We claim that each K_n has empty interior.

Observe that if $w \in K_n$, then $\inf_{k\geq 1}\inf_{x\in\Sigma_2}(S_n w)(x) \geq -n$; for, if not, we could find $N, \varepsilon > 0$, $x_0 \in \Sigma_2$ and open $U \ni x_0$ such that $(S_N w)(x_0) \leq -n - \varepsilon$ for all $x \in U$, and by following the argument in Theorem 1 deduce that any invariant measure μ such that $\mu(U) > 0$ would satisfy $\int w d\mu < 0$, contradicting $w \in W(\Sigma_2)$. Now let $w \in K_n$ and $\varepsilon > 0$, and let $f \in C(\Sigma_2)$ be as above. We have

$$\begin{aligned} \sup_{k\geq 1}\sup_{x\in\Sigma_2} S_k\left(w + \varepsilon f\right)(x) &\geq \inf_{k\geq 1}\inf_{x\in\Sigma_2} S_k w(x) + \varepsilon \sup_{k\geq 1}\sup_{x\in\Sigma_2} S_k f(x) \\ &\geq -n + \varepsilon \sup_{k\geq 1}\sup_{x\in\Sigma_2}(S_k f)(x) = \infty, \end{aligned}$$

which demonstrates that $w + \varepsilon f \notin K_n$, and so K_n has empty interior. The conclusion follows by Baire's Theorem. □

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