The generalized Berger-Wang formula and the spectral radius of linear cocycles

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Abstract. Using multiplicative ergodic theory we prove two formulae describing the relationships between different joint spectral radii for sets of bounded linear operators acting on a Banach space. In particular we recover a formula previously proved by V. S. Shulman and Yu. V. Turovskiĭ using operator-theoretic ideas. As a byproduct of our method we answer a question of J. E. Cohen on the limiting behaviour of the spectral radius of a measurable matrix cocycle.

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1. Introduction and statement of results

Let A be a set of $d \times d$ real matrices. The joint spectral radius of A was defined by G.-C. Rota and W. G. Strang in [**19**] to be the quantity

$$\hat{\varrho}(\mathsf{A}) := \lim_{n\to\infty} \sup \left\{ \|A_{i_n} \cdots A_{i_1}\|^{1/n} \colon A_i \in \mathsf{A} \right\}.$$

The joint spectral radius has since emerged as a useful tool in a number of research areas including the theory of control and stability [**2, 10, 12**], coding theory [**15**], wavelet regularity [**6, 7, 13**], and the study of numerical solutions to ordinary differential equations [**9**]. The following characterisation of the joint spectral radius, which we term the *Berger-Wang formula*, is due to M. A. Berger and Y. Wang [**3**], following a conjecture in [**6**]:

Theorem 1.1. *Let A be a nonempty bounded set of $d \times d$ real matrices. Then*

$$\hat{\varrho}(\mathsf{A}) = \limsup_{n\to\infty} \sup \left\{ \rho(A_{i_n} \cdots A_{i_1})^{1/n} \colon A_i \in \mathsf{A} \right\}.$$

The concept of joint spectral radius generalizes directly to the context of bounded operators on Banach spaces, where it may be used to establish a number of results in invariant subspace theory [**23, 24, 26**]. In this article we use ergodic theory to prove an analogue of Theorem 1.1 for the case in which A is a set of bounded linear operators acting on a Banach space $\mathfrak{X}$. We also obtain a further relationship between two other joint spectral radii which may be defined in this context. In order to state our main results we require the following definitions.

Let $\mathfrak{X}$ be a Banach space and let $B_{\mathfrak{X}}$ denote its unit ball. We define the *Hausdorff measure of noncompactness* of an operator $L \in \mathcal{B}(\mathfrak{X})$, which we denote by

$\|L\|_\chi$, to be the infimum of all positive real numbers ε for which $LB_{\mathfrak{X}}$ admits a finite ε-net. We define a second functional on $\mathcal{B}(\mathfrak{X})$ by $\|L\|_f := \inf\{\|L-K\| : \operatorname{rank} K < \infty\}$. Some useful properties of these two quantities are described in the following proposition.

PROPOSITION 1.2. *Let $\mathfrak{X}$ be a Banach space. Then the functions $\|\cdot\|_\chi, \|\cdot\|_f : \mathcal{B}(\mathfrak{X}) \to \mathbb{R}$ are seminorms and are Lipschitz continuous. For every $L \in \mathcal{B}(\mathfrak{X})$ we have $\|L\|_\chi = 0 \iff L \in \mathcal{K}(\mathfrak{X})$ and $\|L\|_\chi \leq \|L\|_f \leq \|L\|$. The seminorms $\|\cdot\|_\chi$ and $\|\cdot\|_f$ are* submultiplicative: *for every $L_1, L_2 \in \mathcal{B}(\mathfrak{X})$ we have $\|L_1L_2\|_\chi \leq \|L_1\|_\chi\|L_2\|_\chi$ and similarly for $\|\cdot\|_f$.*

The properties listed above for $\|\cdot\|_\chi$ are rather standard (see e.g. [**17**]); the properties of $\|\cdot\|_f$ and the inequality $\|L\|_\chi \leq \|L\|_f$ may easily be deduced by the reader.

Given a Banach space $\mathfrak{X}$ and a bounded set $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$, let us define $\mathsf{A}^n := \{A_{i_n} \cdots A_{i_1} : A_i \in \mathsf{A}\}$ for each $n \in \mathbb{N}$. We consider the following four spectral radii:

$$\hat{\varrho}(\mathsf{A}) := \lim_{n\to\infty} \sup_{A \in \mathsf{A}^n} \|A\|^{1/n}, \qquad \varrho_\chi(\mathsf{A}) := \lim_{n\to\infty} \sup_{A \in \mathsf{A}^n} \|A\|_\chi^{1/n},$$

$$\varrho_f(\mathsf{A}) := \lim_{n\to\infty} \sup_{A \in \mathsf{A}^n} \|A\|_f^{1/n}, \qquad \varrho_r(\mathsf{A}) := \limsup_{n\to\infty} \sup_{A \in \mathsf{A}^n} \rho(A)^{1/n}.$$

By submultiplicativity it follows that the limit in each of the definitions of $\hat{\varrho}$, ϱ_χ and ϱ_f exists and is also equal to the infimum over $n \in \mathbb{N}$ of the same quantity.

In this article we establish the following basic relationships between the spectral radii $\hat{\varrho}$, ϱ_r, ϱ_χ and ϱ_f:

THEOREM 1.3. *Let $\mathfrak{X}$ be a Banach space and let $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$ be precompact and nonempty. Then*

$$\hat{\varrho}(\mathsf{A}) = \max\{\varrho_\chi(\mathsf{A}), \varrho_r(\mathsf{A})\} \tag{1}$$

and

$$\varrho_\chi(\mathsf{A}) = \varrho_f(\mathsf{A}). \tag{2}$$

The equation (1) has been termed the *Generalized Berger-Wang formula* by V. S. Shulman and Yu. V. Turovskiĭ [**24**], who gave a proof conditional on various additional hypotheses. A more general version was established in [**25**], with an unconditional proof recently being given in [**26**]. The relation (2) does not appear to have been known prior to the present article, although a similar relation is known in cases where $\mathfrak{X}$ satisfies a strong version of the compact approximation property [**24**, p.419].

As well as strictly generalising Theorem 1.1, Theorem 1.3 yields simple proofs of several results on invariant subspaces of families of operators. In particular it may be applied to give a short proof of a celebrated theorem of Turovskiĭ [**29**] which states that any semigroup of compact quasinilpotent operators acting on a Banach space has a nontrivial invariant subspace. Proofs via (1) of this and other related results may be found in [**24, 26**].

It should be noted that (1) may fail to hold if we assume only that A is bounded, and also that there exist Banach spaces $\mathfrak{X}$ and precompact sets $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$ such that $\varrho_r(\mathsf{A}) < \varrho_\chi(\mathsf{A})$. Specifically, in the case where $\mathfrak{X}$ is a separable infinite-dimensional Hilbert space, M.-H. Shih *et al.* give an example in [**22**] of a closed bounded noncompact set $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$ which satisfies $\varrho_\chi(\mathsf{A}) = \varrho_r(\mathsf{A}) = 0$ and $\hat{\varrho}(\mathsf{A}) = 1$. Also in the context of a separable Hilbert space, an example was given by P. Rosenthal and A. Sołtysiak in [**18**] of a two-element set $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$ such that $\varrho_r(\mathsf{A}) < \hat{\varrho}(\mathsf{A})$. It is not clear to the author whether (2) can fail to hold when A is bounded but not precompact.

We now describe the ergodic-theoretic results which are used to deduce Theorem 1.3. Let T be a measure-preserving transformation of a probability space $(X, \mathcal{F}, \mu)$ and $\mathfrak{X}$ a Banach space. A *linear cocycle* over T is a measurable function $\mathcal{A}\colon X \times \mathbb{N} \to \mathcal{B}(\mathfrak{X})$ which satisfies the relation

$$\mathcal{A}(x, n+m) = \mathcal{A}(T^m x, n)\mathcal{A}(x, m)$$

for every $n, m \in \mathbb{N}$ and μ-almost-every $x \in X$. If X is a topological space and $\mathcal{F}$ its Borel σ-algebra we say that $\mathcal{A}$ is a continuous linear cocycle if the map $x \mapsto \mathcal{A}(x, k)$ is continuous for every $k \in \mathbb{N}$. When discussing linear cocycles we shall find it useful to adopt the conventions $\log 0 := -\infty$ and $\log^+(x) := \max\{0, \log x\}$ for $x \geq 0$.

Theorem 1.3 is derived from the following technical result:

Theorem 1.4. *Let $T\colon X \to X$ be a homeomorphism of a compact metric space, μ a T-invariant Borel probability measure on X, and $\mathfrak{X}$ a Banach space. Suppose that $\mathcal{A}\colon X \times \mathbb{N} \to \mathcal{B}(\mathfrak{X})$ is a continuous cocycle such that $\mathcal{A}(x, n)$ is an injective operator for every $(x, n) \in X \times \mathbb{N}$. Then there exists a T-invariant Borel set $\Lambda \subseteq X$ satisfying $\mu(\Lambda) = 1$ with the following properties. If $x \in \Lambda$ then the limits*

$$\lambda(x) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(x, n)\|,$$

$$\chi(x) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(x, n)\|_\chi$$

exist,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(x, n)\|_f = \chi(x), \tag{3}$$

and if additionally $\chi(x) < \lambda(x)$ then

$$\limsup_{n\to\infty} \frac{1}{n} \log \rho(\mathcal{A}(x, n)) = \lambda(x). \tag{4}$$

For $d \in \mathbb{N}$ let $\mathrm{Mat}_d(\mathbb{R})$ denote the vector space of all $d \times d$ real matrices. The ideas used to prove Theorem 1.4 may also be applied to obtain the following result which answers a question of J. E. Cohen [**4**, p.329]:

Theorem 1.5. *Let T be a measure-preserving transformation of a probability space $(X, \mathcal{F}, \mu)$ and let $\mathcal{A}\colon X \times \mathbb{N} \to \mathrm{Mat}_d(\mathbb{R})$ be a measurable linear cocycle such that $\int \log^+ \|\mathcal{A}(x, 1)\|\, d\mu(x) < \infty$. Let Z denote the set of all $x \in X$ such that $\lim_{n\to\infty} \|\mathcal{A}(x, n)\|^{1/n}$ exists. Then*

$$\mu\left(\left\{x \in Z\colon \limsup_{n\to\infty} \frac{1}{n} \log \rho(\mathcal{A}(x, n)) = \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(x, n)\|\right\}\right) = 1. \tag{5}$$

A version of Theorem 1.5 was given by A. Avila and J. Bochi [**1**] in the case where T is invertible and $\mathcal{A}(x, 1) \in SL_2(\mathbb{R})$ for every $x \in X$. Avila and Bochi also give an example to show that $\lim_{n\to\infty} \frac{1}{n} \log \rho(\mathcal{A}(x, n))$ can fail to exist μ-a.e. even when the dependence between $\mathcal{A}(x, 1)$ and $\mathcal{A}(T^k x, 1)$ is very weak.

2. Proof of Theorems 1.4 and 1.5

The main ingredient of the proof of Theorem 1.4 is a multiplicative ergodic theorem for Banach spaces which was established by P. Thieullen [**28**], building on earlier work of R. Mañé [**14**] and D. Ruelle [**20**]. In order to state Thieullen's result we require the following definitions.

Given a Banach space $\mathfrak{X}$ we let $\mathcal{G}(\mathfrak{X})$ denote the set of all closed subspaces $F \subseteq \mathfrak{X}$ for which there exists a closed subspace $G \subseteq \mathfrak{X}$ such that $\mathfrak{X} = F \oplus G$. In particular, $\mathcal{G}(\mathfrak{X})$ contains all finite-dimensional subspaces of $\mathfrak{X}$ as well as all closed subspaces of finite codimension. Given any $F_0 \in \mathcal{G}(\mathfrak{X})$ and any $G \in \mathcal{G}(\mathfrak{X})$ such that $F_0 \oplus G = \mathfrak{X}$, define $U_{F_0,G} = \{F \in \mathcal{G}(\mathfrak{X}) \colon F \oplus G = \mathfrak{X}\}$. Define a map $\varphi_{F_0,G} \colon U_{F_0,G} \to \mathcal{B}(F_0, G)$ by identifying each $F \in U_{F_0,G}$ with the unique continuous linear map $F_0 \to G$ whose graph is F; that is, $\varphi_{F_0,G}(F)$ is the restriction to F_0 of the unique projection $\mathfrak{X} \to G$ having image G and kernel F, this projection being continuous by the closed graph theorem. We now define a topology on $\mathcal{G}(\mathfrak{X})$ by declaring each triple $(U_{F_0,G}, \varphi_{F_0,G}, \mathcal{B}(F_0, G))$ to be a chart at F_0. The resulting topology has the property that $F_n \to F$ in $\mathcal{G}(\mathfrak{X})$ if and only if $\varphi_{F,G}(F_n) \to 0 \in \mathcal{B}(F, G)$ for every $G \in \mathcal{G}(\mathfrak{X})$ such that $F \oplus G = \mathfrak{X}$; see [**28**, Appendix B].

Given topological spaces Ω_1, Ω_2 and a Borel measure μ on Ω_1, we shall say that a map $f \colon \Omega_1 \to \Omega_2$ is *μ-continuous* if there exists a sequence of pairwise disjoint compact sets $K_n \subseteq X$ such that $\mu(\bigcup_n K_n) = 1$ and the restriction of f to K_n is continuous for each n.

We now give a statement of Thieullen's theorem, restricted to the special case of a compact metric space X and ergodic measure μ. To simplify the statement let us write $\mathbb{N}_p := \{1, \ldots, p\}$ for $p \in \mathbb{N}$ and $\mathbb{N}_\infty := \mathbb{N}$.

Theorem 2.1. *Let $T \colon X \to X$ be a continuous homeomorphism of a compact metric space, let μ be an ergodic Borel probability measure on X, let $\mathfrak{X}$ be a Banach space, and let $\mathcal{A} \colon X \times \mathbb{N} \to \mathcal{B}(\mathfrak{X})$ be a continuous cocycle. Suppose that $\mathcal{A}(x, n)$ is an injective operator for every $(x, n) \in X \times \mathbb{N}$. By the subadditive ergodic theorem the limits $\chi := \lim \frac{1}{n} \log \|\mathcal{A}(x, n)\|_\chi$ and $\lambda := \lim \frac{1}{n} \log \|\mathcal{A}(x, n)\|$ exist and are constant μ-a.e. Suppose that $\chi < \lambda$. Then there exists a T-invariant Borel set $\Lambda \subseteq X$ which satisifies $\mu(\Lambda) = 1$ such that the following properties hold.*

There exist $p \in \mathbb{N} \cup \{\infty\}$, a sequence of real numbers $\lambda = \lambda_1 > \lambda_2 > \ldots > \chi$ indexed in $\mathbb{N}_p$, and two corresponding sequences of Borel-measurable μ-continuous maps $F_1, F_2, \ldots \colon \Lambda \to \mathcal{G}(\mathfrak{X})$, $G_1, G_2, \ldots, \colon \Lambda \to \mathcal{G}(\mathfrak{X})$ indexed in $\mathbb{N}_p$, such that for every $x \in \Lambda$ and $k \in \mathbb{N}_p$, $F_k(x)$ is finite-dimensional and

$$F_1(x) \oplus \cdots \oplus F_k(x) \oplus G_k(x) = \mathfrak{X},$$

$$\mathcal{A}(x, n)F_k(x) = F_k(T^n x),$$

$$\mathcal{A}(x, n)G_k(x) = G_k(T^n x)$$

for every $n \in \mathbb{N}$. For each $x \in \Lambda$ and $k \in \mathbb{N}_p$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\|\mathcal{A}(x, n)v\|}{\|v\|} = \lambda_k$$

uniformly for $v \in F_k(x) \setminus \{0\}$. Similarly, if $k \in \mathbb{N}$ with $k < p$ then for each $x \in \Lambda$,

$$\lim_{n \to \infty} \frac{1}{n} \log \sup \left\{ \frac{\|\mathcal{A}(x, n)v\|}{\|v\|} \colon v \in G_k(x) \setminus \{0\} \right\} = \lambda_{k+1}. \tag{6}$$

If $k = p < \infty$ *then the limit in* (6) *is instead equal to* χ*. If* $p = \infty$ *then* $\lim_{i\to\infty} \lambda_i = \chi$.

Remark. The statement of uniform convergence in Theorem 2.1 is not announced explicitly in the statement of that theorem in Thieullen's paper. However, the corresponding statement is proved in [**28**, p.68-69].

We may now begin the proof of Theorem 1.4. Let $X, T, \mathfrak{X}$ and $\mathcal{A}$ be as in that theorem. We must show that the set of all $x \in X$ such that

$$\limsup_{n\to\infty} \rho(\mathcal{A}(x,n))^{1/n} = \liminf_{n\to\infty} \|\mathcal{A}(x,n)\|^{1/n} = \limsup_{n\to\infty} \|\mathcal{A}(x,n)\|^{1/n}$$

and

$$\liminf_{n\to\infty} \|\mathcal{A}(x,n)\|_\chi^{1/n} = \limsup_{n\to\infty} \|\mathcal{A}(x,n)\|_f^{1/n},$$

which is clearly Borel, has full measure for every T-invariant measure μ. Using a suitable ergodic decomposition theorem it suffices to prove that this holds in cases where μ is ergodic, which assumption we make for the remainder of the section.

We begin by showing that $\lim \frac{1}{n} \log \|\mathcal{A}(x,n)\|_f = \lim \frac{1}{n} \log \|\mathcal{A}(x,n)\|_\chi$ μ-a.e. Let λ, χ be as above; if $\lambda = \chi$ then the result follows directly from Proposition 1.2, so we assume $\lambda > \chi$. Let $\Lambda \subseteq X$, $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ be as given by Theorem 2.1. Given $\varepsilon > 0$, take $k \geq 1$ such that for every $x \in \Lambda$ the limit in (6) is bounded by $\chi + \varepsilon$. For each $x \in \Lambda$ define $F(x) := F_1(x) \oplus \cdots \oplus F_k(x)$ and let $P(x) \in \mathcal{B}(\mathfrak{X})$ be the unique projection having image $F(x)$ and kernel $G_k(x)$. Note that for each $x \in \Lambda$ the boundedness of $P(x)$ is guaranteed by the closed graph theorem, and that the μ-continuity of the maps $F_1, \ldots, F_k$, G_k implies the μ-continuity of P. Now take any $x \in \Lambda$. Since $P(x)$ has finite rank,

$$\begin{aligned} \|\mathcal{A}(x,n)\|_f &\leq \|\mathcal{A}(x,n) - \mathcal{A}(x,n)P(x)\| \\ &\leq (1 + \|P(x)\|) \sup\left\{\frac{\|\mathcal{A}(x,n)v\|}{\|v\|} : v \in G_k(x) \setminus \{0\}\right\} \end{aligned}$$

so that $\limsup \frac{1}{n} \log \|\mathcal{A}(x,n)\|_f \leq \chi + \varepsilon$ for each $x \in \Lambda$. By Proposition 1.2 we have $\liminf \frac{1}{n} \log \|\mathcal{A}(x,n)\|_f \geq \chi$ and the result follows.

We now proceed to the second part of Theorem 1.4. Our approach is suggested by recent work of B. Kalinin [**11**]; this line of argument is also applied by the author in [**16**]. For each $x \in \Lambda$ we take $V(x) = F_1(x)$ and $W(x) = G_1(x)$, let $P(x) \in \mathcal{B}(\mathfrak{X})$ be the unique projection having image $V(x)$ and kernel $W(x)$, and define $Q(x) = I - P(x)$. Clearly P and Q are well-defined and μ-continuous as before. Let $x \in \Lambda$ and $n \in \mathbb{N}$; if $v \in V(x)$ and $w \in W(x)$ then clearly

$$P(T^n x)\mathcal{A}(x,n)(v + w) = \mathcal{A}(x,n)v = \mathcal{A}(x,n)P(x)(v + w),$$

and since $V(x) \oplus W(x) = \mathfrak{X}$ it follows that $P(T^n x)\mathcal{A}(x,n) = \mathcal{A}(x,n)P(x)$ and $Q(T^n x)\mathcal{A}(x,n) = \mathcal{A}(x,n)Q(x)$. We require the following lemma:

Lemma 2.2. *There exists a set* $\tilde{\Lambda} \subseteq \Lambda$ *with* $T\tilde{\Lambda} \subseteq \tilde{\Lambda}$ *and* $\mu(\tilde{\Lambda}) = 1$ *such that for every* $x \in \tilde{\Lambda}$,

$$\liminf_{n\to\infty} \|P(x) - P(T^n x)\| = 0.$$

Proof. It suffices to show that for each $r > 0$, the set

$$\Lambda_r := \left\{x \in \Lambda \colon \liminf_{n\to\infty} \|P(x) - P(T^n x)\| \leq 2/r\right\}$$

has full measure, since we may then define

$$\tilde{\Lambda} = \bigcap_{n=0}^{\infty} T^{-n}\left(\bigcap_{r=1}^{\infty} \Lambda_r\right)$$

and obtain the desired result.

Let K_n be a sequence of compact subsets of X witnessing the μ-continuity of the map $x \mapsto P(x)$. Define

$$\mathcal{Z} = \bigcup_{n\geq 1} \{P(x) \colon x \in K_n\} \subseteq \mathcal{B}(\mathfrak{X}).$$

Clearly $\mathcal{Z}$ is a countable union of compact sets, hence separable, and $\mu(\{x \in \Lambda \colon P(x) \in \mathcal{Z}\}) = 1$. Let $r > 0$. Since $\mathcal{Z}$ is separable we may choose a sequence $(L_n)_{n\geq 1}$ in $\mathcal{Z}$ such that $\{B_{1/r}(L_n) \colon n \geq 1\}$ covers $\mathcal{Z}$. For each $n > 0$ let $C_{n,r} = \{x \in \Lambda \colon P(x) \in B_{1/r}(L_n)\}$. For each $n > 0$ the Poincare recurrence theorem yields

$$\mu(\{x \in C_{n,r} \colon T^k x \in C_{n,r} \text{ for infinitely many } k \in \mathbb{N}\}) = \mu(C_{n,r})$$

and since $\mu(\bigcup_{n\geq 1} C_{n,r}) = 1$ it follows that $\mu(\Lambda_r) = 1$ as required. □

For each $\delta > 0$ and $x \in \Lambda$ define a closed convex cone in $\mathfrak{X}$ by

$$K(x,\delta) := \left\{u \in \mathfrak{X} \colon \|P(x)u\| \geq \delta^{-1}\|Q(x)u\|\right\}.$$

To prove Theorem 1.4 it suffices to show that each $x \in \tilde{\Lambda}$ has the following two properties: firstly, for every sufficiently small $\varepsilon > 0$ we have

$$\inf_{u\in K(x,1)\setminus\{0\}} \frac{\|\mathcal{A}(x,n)u\|}{\|u\|} \geq e^{n(\lambda-3\varepsilon)} \tag{7}$$

for all sufficiently large $n > 0$; and secondly, for infinitely many $n > 0$ we have $\mathcal{A}(x,n)K(x,1) \subseteq K(x,1)$. To see that this implies (4), note that if $\mathcal{A}(x,n)K(x,1)$ is contained in $K(x,1)$ and (7) holds, then taking any $v \in K(x,1) \setminus \{0\}$ we obtain

$$\frac{1}{n}\log\rho(\mathcal{A}(x,n)) \geq \liminf_{k\to\infty} \frac{1}{nk}\log\left\|(\mathcal{A}(x,n))^k v\right\| \geq \lambda - 3\varepsilon.$$

Given $x \in \tilde{\Lambda}$ it follows that (4) will be satisfied if the above conditions can be met for every $\varepsilon > 0$. We therefore fix $x \in \tilde{\Lambda}$ for the remainder of the proof and proceed to establish these two properties. By Theorem 2.1 we have

$$\liminf_{n\to\infty} \frac{1}{n}\log\inf\left\{\frac{\|\mathcal{A}(x,n)v\|}{\|v\|} \colon v \in V(x)\setminus\{0\}\right\} = \lambda$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\log\sup\left\{\frac{\|\mathcal{A}(x,n)v\|}{\|v\|} \colon v \in W(x)\setminus\{0\}\right\} = \nu$$

for some $\nu < \lambda$. Choose any $\varepsilon > 0$ small enough that $3\varepsilon < \lambda - \nu$. If n is taken large enough we have for each $u \in K(x,1)$

$$\begin{aligned}\|P(T^n x)\mathcal{A}(x,n)u\| = \|\mathcal{A}(x,n)P(x)u\| &\geq e^{n(\lambda-\varepsilon)}\|P(x)u\| \\ &\geq \frac{1}{2}e^{n(\lambda-\varepsilon)}\|u\| \geq e^{n(\lambda-2\varepsilon)}\|u\|\end{aligned}$$

and

$$\|Q(T^n x)\mathcal{A}(x,n)u\| = \|\mathcal{A}(x,n)Q(x)u\| \leq e^{n(\nu+\varepsilon)}\|Q(x)u\| \leq e^{n(\nu+\varepsilon)}\|Q(x)\|.\|u\|$$

where we have used the inequality $\|u\| = \|(P(x) + Q(x))u\| \leq 2\|P(x)u\|$ which holds for all $u \in K(x, 1)$. Combining the above expressions yields

$$\|Q(T^n x)\mathcal{A}(x,n)u\| \leq e^{n(\nu+3\varepsilon-\lambda)}\|Q(x)\|.\|P(T^n x)\mathcal{A}(x,n)u\|$$

for every $u \in K(x, 1)$, from which we conclude that for each $\delta > 0$ we have $\mathcal{A}(x,n)K(x,1) \subseteq K(T^n x, \delta)$ for all large enough n. Additionally we obtain

$$\begin{aligned}\|\mathcal{A}(x,n)u\| &\geq \|P(T^n x)\mathcal{A}(x,n)u\| - \|Q(T^n x)\mathcal{A}(x,n)u\| \\ &\geq \left(e^{n(\lambda-2\varepsilon)} - e^{n(\nu+\varepsilon)}\|Q(x)\|\right)\|u\|\end{aligned}$$

for every $u \in K(x, 1)$, which gives (7) when n is large enough.

To complete the proof we show that for every $\delta \in (0, 1)$ we have $K(T^n x, \delta) \subseteq K(x, 1)$ for infinitely many n. Given $\delta \in (0, 1)$, choose $\kappa > 0$ such that $\delta^{-1} - 2\kappa(1+\delta^{-1}) > 1$. By Lemma 2.2 we have $\|P(x) - P(T^n x)\| < \kappa$ for infinitely many $n > 0$. For each such n we have

$$\begin{aligned}\|P(x)u\| \geq \|P(T^n x)u\| - \kappa\|u\| &\geq \delta^{-1}\|Q(T^n x)u\| - \kappa\|u\| \\ &\geq \delta^{-1}\|Q(x)u\| - \kappa(1+\delta^{-1})\|u\|\end{aligned}$$

for every $u \in K(T^n x, \delta)$, where we have used the relation $\|P(x) - P(T^n x)\| = \|Q(x) - Q(T^n x)\|$ which follows from the definition of Q. If $u \in K(T^n x, \delta) \setminus K(x, 1)$ then additionally $\|u\| < 2\|Q(x)u\|$ and therefore

$$\|P(x)u\| > \left(\delta^{-1} - 2\kappa(1+\delta^{-1})\right)\|Q(x)u\| \geq \|Q(x)u\|$$

contradicting $u \notin K(x, 1)$. We conclude that $K(T^n x, \delta) \setminus K(x, 1) = \emptyset$ and therefore $K(T^n x, \delta) \subseteq K(x, 1)$ as required. The proof of Theorem 1.4 is complete.

The proof of Theorem 1.5 may be undertaken by pursuing *mutatis mutandis* the proof of Theorem 1.4, if we allow the additional assumption that T is invertible. In this case we apply the following result in lieu of Theorem 2.1.

Theorem 2.3. *Let T be an ergodic invertible measure-preserving transformation of a complete probability space $(X, \mathcal{F}, \mu)$, let $\mathcal{A}\colon X \times \mathbb{N} \to \mathrm{Mat}_d(\mathbb{R})$ be a measurable cocycle such that $\int \log^+ \|\mathcal{A}(x,1)\|\, d\mu(x) < \infty$, and define the quantity $\lambda := \inf_{n\geq 1} \frac{1}{n}\int \log\|\mathcal{A}(x,n)\|\, d\mu(x)$. Then there exists a measurable T-invariant set $\Lambda \subseteq X$ satisfying $\mu(\Lambda) = 1$ with the following properties. There exists an integer $p \in \{1, \ldots, d\}$, a finite sequence $\lambda = \lambda_1 > \ldots > \lambda_p \geq -\infty$, and a corresponding sequence of measurable functions $F_1, \ldots, F_p$ from Λ into the Grassmannian of $\mathbb{R}^d$, such that for every $x \in \Lambda$ we have $F_1(x) \oplus \cdots \oplus F_p(x) = \mathbb{R}^d$, $\mathcal{A}(x,n)F_i(x) \subseteq F_i(T^n x)$ for each $1 \leq i \leq p$ and $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|\mathcal{A}(x,n)v\|}{\|v\|} = \lambda_i$$

uniformly for $v \in F_i(x) \setminus \{0\}$.

For a proof see [**8**]. The part of the statement dealing with uniform convergence is not declared in a completely explicit fashion in that article but features clearly in the proof. Since $\mathrm{Mat}_d(\mathbb{R})$ is separable, a suitable analogue of Lemma 2.2 may be proved easily without the additional requirement of μ-continuity.

To complete the proof of Theorem 1.5 it remains to show that its result may be extended from the case of invertible T to the general case. Given a measure-preserving transformation T of a probability space $(X, \mathcal{F}, \mu)$, recall from e.g. [**5**] that there exist an invertible transformation $\hat{T}$ of a probability space $(\hat{X}, \hat{\mathcal{F}}, \hat{\mu})$ and a measurable map $\pi\colon \hat{X} \to X$ such that $\pi^*\hat{\mu} = \mu$ and $T \circ \pi = \pi \circ \hat{T}$ $\hat{\mu}$-a.e. Now, given a cocycle $\mathcal{A}\colon X \times \mathbb{N} \to \mathrm{Mat}_d(\mathbb{R})$ which satisfies the conditions of Theorem 1.5, note that the function $\hat{\mathcal{A}}\colon \hat{X} \times \mathbb{N} \to \mathrm{Mat}_d(\mathbb{R})$ defined by $\hat{\mathcal{A}}(x,n) := \mathcal{A}(\pi x, n)$ is a measurable cocycle with respect to $\hat{T}$. Since $\int_X \log^+ \|\mathcal{A}(x,1)\| d\mu = \int_{\hat{X}} \log^+ \|\hat{\mathcal{A}}(x,1)\| d\hat{\mu}(x)$ by construction, $\hat{\mathcal{A}}$ meets the desired integrability condition and we obtain

$$\mu\left(\left\{x \in X\colon \limsup_{n\to\infty} \rho(\mathcal{A}(x,n))^{1/n} = \lim_{n\to\infty} \|\mathcal{A}(x,n)\|^{1/n}\right\}\right)$$
$$= \hat{\mu}\left(\left\{x \in \hat{X}\colon \limsup_{n\to\infty} \rho(\hat{\mathcal{A}}(x,n))^{1/n} = \lim_{n\to\infty} \|\hat{\mathcal{A}}(x,n)\|^{1/n}\right\}\right) = 1$$

by applying Theorem 1.5 in the invertible case.

3. Proof of Theorem 1.3

To begin the proof, we claim that it is sufficient to demonstrate Theorem 1.3 under the additional hypotheses that A is compact and consists solely of injective elements of $\mathcal{B}(\mathfrak{X})$. We shall first show that if Theorem 1.3 holds for compact sets of bounded operators then it also must hold for *precompact* sets of bounded operators. To see this, fix a precompact set $\mathsf{A} \subseteq \mathcal{B}(\mathfrak{X})$ and suppose that $\hat{\varrho}(\overline{\mathsf{A}}) = \max\{\varrho_\chi(\overline{\mathsf{A}}), \varrho_r(\overline{\mathsf{A}})\}$ and $\varrho_f(\overline{\mathsf{A}}) = \varrho_\chi(\overline{\mathsf{A}})$. It follows from Proposition 1.2 that the maps $\|\cdot\|_\chi, \|\cdot\|_f\colon \mathcal{B}(\mathfrak{X}) \to \mathbb{R}$ are continuous, whereupon a simple inspection of the definitions yields $\hat{\varrho}(\mathsf{A}) = \hat{\varrho}(\overline{\mathsf{A}})$, $\varrho_\chi(\mathsf{A}) = \varrho_\chi(\overline{\mathsf{A}})$ and $\varrho_f(\mathsf{A}) = \varrho_f(\overline{\mathsf{A}})$ so that in particular (2) holds. If $\hat{\varrho}(\overline{\mathsf{A}}) = \varrho_\chi(\overline{\mathsf{A}})$ then (1) is clearly satisfied and the argument is complete. If otherwise, given any small enough $\varepsilon > 0$ there exist infinitely many $n \in \mathbb{N}$ such that

$$\sup_{A \in \overline{\mathsf{A}}^n} \rho(A)^{1/n} > \hat{\varrho}(\overline{\mathsf{A}}) - \varepsilon > \sup_{A \in \overline{\mathsf{A}}^n} \|A\|_\chi^{1/n}.$$

Given such an n, choose any $B \in \overline{\mathsf{A}}^n$ with $\rho(B)^{1/n} > \hat{\varrho}(\overline{\mathsf{A}}) - \varepsilon$. Since $\rho(B) > \|B\|_\chi$ the operator B has essential spectral radius strictly smaller than its spectral radius, and it follows easily that B is a point of continuity of the spectral radius functional $\rho\colon \mathcal{B}(\mathfrak{X}) \to \mathbb{R}$ (see [**24**, Lemma 9.3] for details). Consequently we have

$$\sup_{A \in \mathsf{A}^n} \rho(A)^{1/n} \geq \rho(B)^{1/n} > \hat{\varrho}(\overline{\mathsf{A}}) - \varepsilon = \hat{\varrho}(\mathsf{A}) - \varepsilon,$$

and since this holds for infinitely many n we deduce that $\varrho_r(\mathsf{A}) \geq \hat{\varrho}(\mathsf{A}) - \varepsilon$. We conclude that $\varrho_r(\mathsf{A}) = \hat{\varrho}(\mathsf{A})$ and (1) is satisfied.

We next show that if Theorem 1.3 holds for compact sets of injective bounded operators then it must hold for all compact sets of bounded operators. We apply a trick used by R. Mañé [**14**] and P. Thieullen [**28**] which resembles the construction of the invertible natural extension of a dynamical system. Define a new Banach space $(\mathfrak{X}_\infty, \|\cdot\|)$ by $\mathfrak{X}_\infty := \mathfrak{X}^\mathbb{N}$ and $\|(v_i)_{i\in\mathbb{N}}\| = \sup\{\|v_i\|\colon i \in \mathbb{N}\}$. Let $(\alpha_i)_{i\in\mathbb{N}}$ be a strictly decreasing sequence in $(0,1]$ with the property that for any subadditive sequence $(a_n)_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{1}{n} \max_{0\leq k\leq n} \left(a_{n-k} + \sum_{i=0}^{k} \log \alpha_i\right).$$

(The existence of such a sequence was proved in [**28**]). Define a map $\mathfrak{E}\colon \mathcal{B}(\mathfrak{X}) \to \mathcal{B}(\mathfrak{X}_\infty)$ by $\mathfrak{E}(L)v_1 = Lv_1$, $\mathfrak{E}(L)v_{i+1} = \alpha_i v_i$. Clearly $\mathfrak{E}(L)$ is an injective operator for any $L \in \mathcal{B}(\mathfrak{X})$. The reader may easily verify that for each $n \in \mathbb{N}$ and $L_1, \ldots, L_n \in \mathcal{B}(\mathfrak{X})$,

$$\|\mathfrak{E}(L_n)\cdots\mathfrak{E}(L_1)\| = \max_{0\leq k\leq n}\left(\|L_{n-k}\cdots L_1\|.\prod_{i=0}^{k}\alpha_i\right)$$

with the same relation holding for the seminorms $\|\cdot\|_\chi$ and $\|\cdot\|_f$. As a particular consequence it follows that $\rho(\mathfrak{E}(L_n)\cdots\mathfrak{E}(L_1)) = \rho(L_n\cdots L_1)$ for any $L_1,\ldots,L_n$ and hence $\varrho_r(\mathfrak{E}(\mathsf{A})) = \varrho_r(\mathsf{A})$. Since for each $n \in \mathbb{N}$

$$\sup_{A\in\mathfrak{E}(\mathsf{A})^n}\|A\|^{1/n} = \max_{0\leq k\leq n}\left(\sup_{A\in\mathsf{A}^{n-k}}\|A\|.\prod_{i=0}^{k}\alpha_i\right)^{1/n}$$

and similarly for $\|\cdot\|_\chi$ and $\|\cdot\|_f$, we conclude that $\hat{\varrho}(\mathsf{A}) = \hat{\varrho}(\mathfrak{E}(\mathsf{A}))$, $\varrho_\chi(\mathsf{A}) = \varrho_\chi(\mathfrak{E}(\mathsf{A}))$ and $\varrho_f(\mathsf{A}) = \varrho_f(\mathfrak{E}(\mathsf{A}))$. Since the map $\mathfrak{E}$ is clearly continuous, $\mathfrak{E}(\mathsf{A})$ is a compact subset of $\mathcal{B}(\mathfrak{X}_\infty)$, and so if the conclusion of Theorem 1.3 is valid for the compact set of injective operators $\mathfrak{E}(\mathsf{A})$ then it must be valid for A also.

For the remainder of this section, therefore, we shall assume that $\mathsf{A} \subset \mathcal{B}(\mathfrak{X})$ is a compact nonempty set of injective bounded operators. Without loss of generality we shall assume $\hat{\varrho}(\mathsf{A}) > \varrho_\chi(\mathsf{A}) \geq 0$, since if this relation does not hold then $\hat{\varrho}(\mathsf{A}) = \varrho_f(\mathsf{A}) = \varrho_\chi(\mathsf{A})$ by Proposition 1.2, and Theorem 1.3 thus holds trivially.

The following theorem on subadditive function sequences derives from a theorem of S. J. Schreiber [**21**]. A similar result was also given by R. Sturman and J. Stark independently of Schreiber's work [**27**]. A complete proof may be found in [**16**].

Theorem 3.1. *Let $T\colon X \to X$ be a continuous transformation of a compact metric space, and let $\mathcal{M}_T$ be the set of all T-invariant Borel probability measures on X. Let $(f_n)_{n\geq 1}$ be a sequence of upper semi-continuous functions $f_i\colon X \to \mathbb{R}\cup\{-\infty\}$ such that $f_{n+m}(x) \leq f_n(T^m x) + f_m(x)$ for every $x \in X$ and $n, m \in \mathbb{N}$. Then*

$$\lim_{n\to\infty}\sup_{x\in X}\frac{1}{n}f_n(x) = \sup_{\mu\in\mathcal{M}_T}\inf_{n\geq 1}\frac{1}{n}\int f_n\,d\mu.$$

Define $X = \mathsf{A}^{\mathbb{Z}}$ and equip this set with the product topology under which it is compact and metrisable. Define a homeomorphism $T\colon X \to X$ by $T[(A_i)_{i\in\mathbb{Z}}] = (A_{i+1})_{i\in\mathbb{Z}}$ and a continuous map $\Pi\colon X \to \mathcal{B}(\mathfrak{X})$ by $\Pi[(A_i)_{i\in\mathbb{Z}}] = A_1$. Define a continuous cocycle $\mathcal{A}\colon X\times\mathbb{N} \to \mathcal{B}(\mathfrak{X})$ by setting $\mathcal{A}(x) = \Pi(T^{n-1}x)\cdots\Pi(x)$ for each $x \in X$ and $n \geq 1$, and let $\mathcal{M}_T$ denote the set of all T-invariant Borel probability measures on X, which is nonempty by the Krylov-Bogolioubov Theorem. We have

$$\hat{\varrho}(\mathsf{A}) = \lim_{n\to\infty}\sup_{x\in X}\|\mathcal{A}(x,n)\|^{1/n}, \qquad \varrho_\chi(\mathsf{A}) = \lim_{n\to\infty}\sup_{x\in X}\|\mathcal{A}(x,n)\|_\chi^{1/n},$$

$$\varrho_f(\mathsf{A}) = \lim_{n\to\infty}\sup_{x\in X}\|\mathcal{A}(x,n)\|_f^{1/n}, \qquad \varrho_r(\mathsf{A}) = \limsup_{n\to\infty}\sup_{x\in X}\rho(\mathcal{A}(x,n))^{1/n}$$

directly from the definitions. Applying Theorem 3.1 with $f_n(x) := \log\|\mathcal{A}(x,n)\|$ we deduce

$$\log\hat{\varrho}(\mathsf{A}) = \sup_{\mu\in\mathcal{M}_T}\inf_{n\geq 1}\frac{1}{n}\int\log\|\mathcal{A}(x,n)\|\,d\mu(x).$$

By [**16**, Lemma 3.5] we may choose $\mu \in \mathcal{M}_T$ such that

$$\inf_{n\geq 1} \frac{1}{n}\int \log \|\mathcal{A}(x,n)\|\, d\mu(x) = \log \hat{\varrho}(\mathsf{A}) > \log \varrho_\chi(\mathsf{A}).$$

Let Λ, λ, χ be as in Theorem 1.4. By the subadditive ergodic theorem we have $\int \lambda\, d\mu = \log \hat{\varrho}(\mathsf{A})$ and thus there is a positive measure set $Z \subseteq \Lambda$ such that every $z \in Z$ satisfies

$$\lambda(z) \geq \log \hat{\varrho}(\mathsf{A}) > \log \varrho_\chi(\mathsf{A}) \geq \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(z,n)\|_\chi = \chi(z).$$

Applying Theorem 1.4 we deduce that every $z \in Z$ satisfies

$$\limsup_{n\to\infty} \frac{1}{n} \log \rho(\mathcal{A}(z,n)) = \lambda(z) \geq \log \hat{\varrho}(\mathsf{A})$$

and thus $\log \varrho_r(\mathsf{A}) \geq \log \hat{\varrho}(\mathsf{A})$. Since clearly $\varrho_r(\mathsf{A}) \leq \hat{\varrho}(\mathsf{A})$ we conclude that $\hat{\varrho}(\mathsf{A}) = \varrho_r(\mathsf{A})$ which yields (1).

By Proposition 1.2 we have $\varrho_f(\mathsf{A}) \geq \varrho_\chi(\mathsf{A})$. Suppose $\varrho_f(\mathsf{A}) > \varrho_\chi(\mathsf{A})$. Applying Theorem 3.1 with $f_n(x) := \log \|\mathcal{A}(x,n)\|_f$ we deduce that there exists $\mu \in \mathcal{M}_T$ for which

$$\inf_{n\geq 1} \frac{1}{n}\int \log \|\mathcal{A}(x,n)\|_f\, d\mu(x) > \varrho_\chi(\mathsf{A}).$$

Via Proposition 1.2 this implies

$$\inf_{n\geq 1} \frac{1}{n}\int \log \|\mathcal{A}(x,n)\|\, d\mu(x) > \inf_{n\geq 1} \frac{1}{n}\int \log \|\mathcal{A}(x,n)\|_\chi\, d\mu(x)$$

and so Theorem 1.4 is applicable. Applying the subadditive ergodic theorem again we deduce that there is a positive-measure set $Z \subseteq \Lambda$ such that every $z \in Z$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(z,n)\|_f > \varrho_\chi(\mathsf{A}) \geq \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(z,n)\|_\chi = \chi(z)$$

contradicting (3). We conclude that $\varrho_f(\mathsf{A}) = \varrho_\chi(\mathsf{A})$ and the proof is complete.

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