# MAXIMIZING MEASURES OF GENERIC HÖLDER FUNCTIONS HAVE ZERO ENTROPY

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Abstract. We prove that for a generic real-valued Hölder continuous function $f$ on a subshift of finite type, every shift-invariant probability measure which maximises the integral of $f$ must have zero entropy. An immediate corollary is that zero-temperature limits of equilibrium states of certain one-dimensional lattice systems generically have zero entropy. We prove an analogous statement for generic Lipschitz observations of expanding maps of the circle.

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## 1. Introduction and statement of theorems

Let $T\colon X \to X$ be a dynamical system, where $X$ is a compact metric space and $T$ is a continuous surjection. We consider the following question: given a continuous real-valued function $f\colon X \to \mathbb{R}$, for which $x \in X$ is the maximum ergodic average

$$\beta(f) := \sup_{x\in X} \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x), \tag{1}$$

attained? An alternative formulation is as follows. Writing $\mathcal{M}_T$ for the set of all $T$-invariant Borel probability measures on $X$, one may easily show that

$$\beta(f) = \sup\left\{\int f\,d\mu\colon \mu \in \mathcal{M}_T\right\} \tag{2}$$

and that there exists at least one $\mu \in \mathcal{M}_T$ such that $\int f\,d\mu = \beta(f)$. The Birkhoff ergodic theorem then guarantees the existence of at least one point $x \in X$ for which the supremum (1) is attained. The problem of finding recurrent *optimal orbits* for $f$ in the sense of finding an $x$ which realises the supremum (1), may thus be reduced to the problem of finding those measures $\mu \in \mathcal{M}_T$ which attain the supremum (2). We define a *maximising measure* of $f\colon X \to \mathbb{R}$ to be a measure $\mu \in \mathcal{M}_T$ such that $\int f\,d\mu = \beta(f)$, and denote the set of maximising measures of $f$ by $\mathcal{M}_{\max}(f)$. Interest in optimal orbits and maximising measures has arisen independently from various topics, including the optimal control of hyperbolic dynamical systems [10, 19], the study of Lyapunov exponents [4, 7, 10], and abstract questions regarding the geometry of the set of invariant measures of an expanding map [1, 11]. Maximising measures also occur as zero-temperature limits of Gibbs equilibrium states [5, 13].

A particular research goal is to show that when the dynamical system $(T, X)$ is expanding, hyperbolic, or a subshift of finite type, and $f$ is sufficiently regular, the maximising measures of $f$ are 'typically' supported on a periodic orbit of $T$. Formally, we make the following conjecture:

**Conjecture 1.** *Let $\sigma\colon \Sigma_A \to \Sigma_A$ be a subshift of finite type, and let $H_\theta$ be the Banach space of $\theta$-Hölder continuous functions $\Sigma_A \to \mathbb{R}$, where $0 < \theta < 1$ (see §2 for precise definitions). Then there is an open and dense set $U \subset H_\theta$ such that every $f \in U$ has a unique maximising measure, and this measure is supported on a periodic orbit of $\sigma$. Moreover, every $f \in U$ admits an open neighbourhood $V \subset U$ such that $\mathcal{M}_{\max}(g) = \mathcal{M}_{\max}(f)$ for every $g \in V$.*

A version of this conjecture was stated by G. Yuan and B. R. Hunt in the related context of hyperbolic dynamical systems on compact manifolds [19]. Some related questions and conjectures occur in [2, 7, 10, 12].

We now briefly describe some partial results towards Conjecture 1. Firstly, G. Contreras, A. O. Lopes and P. Thieullen show in [7] that if $\mu \in \mathcal{M}_\sigma$ is supported on a periodic orbit, then the set of all $f \in H_\theta$ such that $\mathcal{M}_{\max}(f) = \{\mu\}$ is open in $H_\theta$. Conversely, Yuan and Hunt show in [19] that if $U \subset H_\theta$ is open and $\mu \in \bigcap_{f\in U} \mathcal{M}_{\max}(f)$, then $\mu$ is supported on a periodic orbit of $\sigma$. An elegant and general second proof of Yuan and Hunt's result is given in [3]. It remains an open problem to show that the set of functions $f \in H_\theta$ admitting a maximising measure supported on a periodic orbit is dense in $H_\theta$.

The objective of the present note is to prove the following weaked version of Conjecture 1:

**Theorem 1.** *Let $\sigma\colon \Sigma_A \to \Sigma_A$ be a subshift of finite type, and let $H_\theta$ be the Banach space of $\theta$-Hölder continuous real functions on $\Sigma$. Then there is a dense $G_\delta$ set $Z \subset H_\theta$ such that for every $f \in Z$, $\mathcal{M}_{\max}(f)$ is a singleton set whose only element has zero metric entropy.*

Examples of $f \in H_\theta$ such that $\mathcal{M}_{\max}(f)$ includes measures of positive entropy may be constructed as follows. If $K \subset \Sigma_A$ is a $\sigma$-invariant compact set which supports a single ergodic measure $\mu_K$, defining $f(x) = -d_\theta(x, K)$ yields $f \in H_\theta$ and $\mathcal{M}_{\max}(f) = \{\mu_K\}$. Constructions of such sets $K$ for which $h_{top}(\mu_K) > 0$ may be found in e.g. [9].

We remark that in the unpublished manuscript [8], J.-P. Conze and Y. Guivarc'h prove a version of Theorem 1 in which $H_\theta$ is replaced with the Banach space of functions $f$ with summable variations. Their result makes use of the fact that locally constant functions are dense in this Banach space, a property not enjoyed by the space $H_\theta$.

Theorem 1 admits the following interpretation in terms of thermodynamic formalism. Recall that for a continuous function $f\colon \Sigma_A \to \mathbb{R}$, the *pressure* of $f$ is defined to be

$$P(f) = \sup\left\{\int f\,d\mu + h(\mu)\colon \mu \in \mathcal{M}_\sigma\right\},$$

where $h(\mu)$ is the entropy of $\mu$ with respect to $\sigma$. If $f$ is Hölder continuous and the shift $\sigma$ is topologically mixing, then $f$ admits a unique *equilibrium state* $\mu_f \in \mathcal{M}_\sigma$, which is defined to be the unique measure such that $P(f) = \int f\,d\mu_f + h(\mu_f)$ (see e.g. [15, 16]). Introducing a real parameter $\lambda$, we consider the family of measures $\{\mu_{\lambda f}\}_{\lambda\geq 1}$. The limit $\lambda \to +\infty$ is termed the *zero-temperature limit* of the measures $\mu_{\lambda f}$ ([5, 13, 14]). We have

**Corollary 1.1.** *Let $\sigma\colon \Sigma_A \to \Sigma_A$ be a topologically mixing subshift of finite type, and let $H_\theta$ be as in Theorem 1. Then there is a dense $G_\delta$ set $Z \subset H_\theta$ such that for every $f \in Z$, the weak-* limit $\lim_{\lambda\to\infty} \mu_{\lambda f}$ exists and has zero entropy.*

*Proof.* Since $\mathcal{M}_\sigma$ is compact in the weak-* topology, it suffices to choose $Z$ such that for each $f \in Z$, the family $\{\mu_{\lambda f}\}_{\lambda \geq 1}$ has at most one weak-* accumulation point in the limit $\lambda \to \infty$. It follows easily from the definitions of $P(\lambda f)$ and $\mu_{\lambda f}$ that any accumulation point of $\mu_{\lambda f}$ must lie in $\mathcal{M}_{\max}(f)$ (see e.g. [7]) and so we may let $Z$ be as in Theorem 1.

□

It is currently an open question whether the family $\{\mu_{\lambda f}\}_{\lambda \geq 1}$ converges in the limit $\lambda \to \infty$ for *every* $f \in H_\theta$. Some partial results may be found in [5, 14].

We also prove a result analogous to Theorem 1 in the context of expanding maps of the circle. Given a $C^2$ expanding map $T$ of the circle $S^1$, let $\mathcal{M}_T$ be the set of all $T$-invariant probability measures on $S^1$, and let $C_{Lip}(S^1)$ be the set of all Lipschitz continuous real-valued functions on $S^1$ equipped with its usual structure as a Banach space. For each $f \in C_{Lip}(S^1)$ we may define $\mathcal{M}_{\max}(f)$ as before. We give an outline of the proof of the following:

**Theorem 2.** *Let $T\colon S^1 \to S^1$ be a $C^2$ expanding map of the circle. Then there is a dense $G_\delta$ set $Z \subset C_{Lip}(S^1)$ such that for every $f \in Z$, $\mathcal{M}_{\max}(f)$ is a singleton set whose only element has zero metric entropy.*

The structure of this note is as follows. In §2 we give the definition of a subshift of finite type and of the spaces $H_\theta$, and state some preliminary lemmas. In §3 we give the proof of Theorem 1. Finally, in §4 we outline the necessary changes to the proof of Theorem 1 which are required to prove Theorem 2.

## 2. Definitions and preliminary results

Let $A$ be a $\ell \times \ell$ matrix with entries in $\{0, 1\}$. We define the (one-sided) subshift of finite type associated to $A$ to be the set

$$\Sigma_A = \left\{x = (x_i)_{i \in \mathbb{N}} : x_i \in \{1, 2, \ldots, \ell\} \text{ and } A(x_i, x_{i+1}) = 1 \text{ for all } i \geq 1\right\}$$

together with the shift map $\sigma\colon \Sigma_A \to \Sigma_A$ defined by

$$\sigma\left[(x_i)_{i \geq 1}\right] = (x_{i+1})_{i \geq 1}.$$

For each $0 < \theta < 1$ we define a metric on $\Sigma_A$ by

$$d_\theta\left((x_i)_{i \geq 1}, (y_i)_{i \geq 1}\right) = \begin{cases} \theta^{\inf\{i\colon x_i \neq y_i\}} & \text{when } x \neq y \\ 0 & \text{when } x = y. \end{cases}$$

We say that $f\colon \Sigma_A \to \mathbb{R}$ is $\theta$-Hölder continuous if it is Lipschitz continuous with respect to the metric $d_\theta$. We denote the space of $\theta$-Hölder continuous functions on $\Sigma_A$ by $H_\theta$. If we define

$$\|f\|_\theta = \sup_{x \in \Sigma_A} |f(x)| + \sup_{\substack{x, y \in \Sigma_A \\ x \neq y}} \frac{|f(x) - f(y)|}{d_\theta(x, y)}$$

for each $f \in H_\theta$, then $(H_\theta, \|\cdot\|_\theta)$ is a Banach space.

We denote by $\mathcal{M}_\sigma$ the set of all $\sigma$-invariant Borel probability measures on $\Sigma_A$. For each $f\colon \Sigma_A \to \mathbb{R}$, we let $\beta(f) = \sup_{\mu \in \mathcal{M}_\sigma} \int f\, d\mu$ and $\mathcal{M}_{\max}(f) = \left\{\mu \in \mathcal{M}_\sigma\colon \int f\, d\mu = \beta(f)\right\}$ as in §1. For every $\mu \in \mathcal{M}_\sigma$, we denote the metric entropy of $\mu$ with respect to $\sigma$ by $h(\mu)$.

We now summarise some already-known results in a manner convenient for the proof of Theorem 1.

**Lemma 2.1.** *Let $a_1, \ldots, a_n$ be nonnegative real numbers, and let $A = \sum_{i=1}^n a_i \geq 0$. Then*

$$\sum_{i=1}^{n} -a_i \log a_i \leq 1 + A \log n$$

*where we use the convention $0 \log 0 = 0$.*

*Proof.* Applying Jensen's inequality to the concave function $x \mapsto -x \log x$ yields

$$\frac{1}{n}\sum_{i=1}^{n} -a_i \log a_i \leq -\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) \log \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) = -\frac{A}{n}\log A + \frac{A}{n}\log n$$

from which the result follows. □

**Lemma 2.2.** *Let $f \in H_\theta$, and suppose that $\mathcal{M}_{\max}(f) = \{\mu\}$ for some $\mu \in \mathcal{M}_\sigma$. Then there is $C > 0$ such that for every $\nu \in \mathcal{M}_\sigma$,*

$$\beta(f) - C \int d_\theta(x, K)\, d\nu \leq \int f\, d\nu$$

*where $K = \operatorname{supp} \mu$.*

*Proof.* By a well-known lemma (see e.g. [2, 7, 17]) there exists $g \in H_\theta$ such that $f - g \circ \sigma + g \leq \beta(f)$. Define $\tilde{f} = f - g \circ \sigma + g$. Since $\tilde{f}$ is continuous, $\int \tilde{f}\, d\mu = \int f\, d\mu = \beta(f)$ and $\tilde{f} \leq \beta(f)$, it follows that $\tilde{f}(x) = \beta(f)$ for every $x \in K = \operatorname{supp} \mu$. Let $C = \|\tilde{f}\|_\theta$. Given $x \in \Sigma_A$, let $z \in K$ such that $d_\theta(x, z) = d_\theta(x, K)$; we then have

$$\tilde{f}(x) \geq \tilde{f}(z) - C d_\theta(x, z) = \beta(f) - C d_\theta(x, K)$$

from which the result follows. □

For a proof of the following result see [7]. A particularly general version may be found in [12].

**Proposition 2.3.** *Let $T\colon X \to X$ be a continuous surjection of a compact metric space, and let $\mathfrak{X}$ be a Banach space which embeds continuously in $C_0(X; \mathbb{R})$ with dense image. Then the set of all $f \in \mathfrak{X}$ such that $\mathcal{M}_{\max}(f)$ is a singleton set is a dense $G_\delta$ subset of $\mathfrak{X}$.*

## 3. Proof of Theorem 1

For each $p \geq 1$, we let $\mathcal{M}_\sigma^p$ denote the set of all ergodic elements of $\mathcal{M}_\sigma$ which are supported on a periodic orbit of prime period less than or equal to $p$. In the sequel we will identify a periodic orbit $\{z, \sigma z, \ldots, \sigma^{p-1} z\}$ of $\sigma$ with its corresponding invariant measure $\mu = \frac{1}{p}\sum_{j=0}^{p-1} \delta_{\sigma^j z} \in \mathcal{M}_\sigma^p$, and refer to a measure $\mu$ as 'being' a periodic orbit. We fix $\theta \in (0, 1)$ throughout this section.

For each $\gamma > 0$, define

$$\mathcal{E}_\gamma = \{f \in H_\theta \colon h(\mu) < 2\gamma h_{top}(\sigma) \text{ for every } \mu \in \mathcal{M}_{\max}(f)\}.$$

To prove the theorem it suffices to show that $\mathcal{E}_\gamma$ is open and dense in $H_\theta$ for every $\gamma > 0$, and then apply Baire's theorem and Proposition 2.3. To this end we fix $\gamma > 0$ for the remainder of the proof.

**Step 1.** We begin by showing that each $\mathcal{E}_\gamma$ is open. Let $f \in H_\theta$, and suppose that $f_n \to f$ in $H_\theta$. If $f_n \in H_\theta \setminus \mathcal{E}_\gamma$ for every $n > 0$, then for each $n$ we may choose

$\nu^n \in \mathcal{M}_{\max}(f_n)$ such that $h(\nu^n) \geq 2\gamma h_{top}(\sigma)$. Taking a subsequence if necessary, we may assume that $\nu^n \to \nu \in \mathcal{M}_\sigma$. For any $\mu \in \mathcal{M}_\sigma$, we have for each $n > 0$

$$\int f\, d\mu - |f - f_n|_\infty \leq \int f_n\, d\mu \leq \int f_n\, d\nu^n \leq \int f\, d\nu^n + |f_n - f|_\infty$$

and therefore $\int f\, d\mu \leq \int f\, d\nu$. It follows that $\nu \in \mathcal{M}_{\max}(f)$. Since $\nu^n \to \nu$ and the entropy map $m \mapsto h(m)$ is upper semi-continuous (see e.g. [18]) we have $h(\nu) \geq 2\gamma h_{top}(\sigma)$ and therefore $f \in H_\theta \setminus \mathcal{E}_\gamma$. We conclude that $H_\theta \setminus \mathcal{E}_\gamma$ is closed and therefore $\mathcal{E}_\gamma$ is open.

**Step 2.** We must show that $\mathcal{E}_\gamma$ intersects every nonempty open subset of $H_\theta$; this takes up the bulk of the proof. We begin by reducing the problem slightly.

Let $U \subset H_\theta$ be open and nonempty; by Proposition 2.3, there exists $f \in U$ such that $\mathcal{M}_{\max}(f)$ consists of a single element. If this element is a periodic orbit, then $f \in \mathcal{E}_\gamma \cap U$ and we are done. Otherwise, by Lemma 2.2 there exists a real number $C > 0$ and a compact invariant set $K$ such that for every $\mu \in \mathcal{M}_\sigma$,

$$\beta(f) - C\int d_\theta(x, K)\, d\mu \leq \int f\, d\mu \tag{3}$$

and such that $K$ does not contain a periodic orbit.

Let $\varepsilon > 0$ be small enough that $f + g \in U$ whenever $\|g\|_\theta \leq 2\varepsilon$. We will construct a sequence of approximating functions $f_n$ such that $f_n \in U \cap \mathcal{E}_\gamma$ for large enough $n$. In the next two steps we choose a sequence of periodic orbits which will be used in the construction.

**Step 3.** We make the following claim: there exists a sequence of integers $(m_n)_{n\geq 1}$ and a sequence of periodic orbits $\mu_n \in \mathcal{M}_\sigma^n$ such that

$$\int d_\theta(x, K)\, d\mu_n = o(\theta^{m_n})$$

and

$$\lim_{n\to\infty} \frac{\log n}{m_n} = 0.$$

*Proof of claim.* A theorem of X. Bressaud and A. Quas[1] [6, Theorem 2] states that for every $k > 0$,

$$\lim_{n\to\infty} n^k \left( \inf_{\mu \in \mathcal{M}_\sigma^n} \int d_\theta(x, K)\, d\mu \right) = 0.$$

Thus there exists a sequence of periodic orbits $\mu_n$, with each $\mu_n \in \mathcal{M}_\sigma^n$, such that for every $k > 0$

$$\lim_{n\to\infty} n^k \int d_\theta(x, K)\, d\mu_n = 0.$$

Define a sequence of real numbers $r_n$ by

$$r_n = \log_\theta \left( \int d_\theta(x, K)\, d\mu_n \right).$$

Since

$$\theta^{r_n} \leq n^k \theta^{r_n} \leq 1 \iff 0 \geq \frac{\log_\theta n}{r_n} \geq -\frac{1}{k}$$

[1]Bressaud and Quas' theorem is stated under the requirement that $\Sigma$ be a full shift, but their proof may be easily adapted to the case of a subshift of finite type.

we deduce that $r_n^{-1}\log n \to 0$. We therefore define $m_n = \lfloor \frac{1}{2} r_n \rfloor$ so as to yield $m_n^{-1}\log n \to 0$ and

$$\int d_\theta(x,K)\,d\mu_n = \theta^{r_n} \leq \theta^{m_n+\frac{1}{2}r_n} = o(\theta^{m_n})$$

as required.

**Step 4.** For each $n \geq 1$, define $L_n = \operatorname{supp}\mu_n$. We make the following second claim concerning the periodic orbits $\mu_n$: there exists $N_\gamma > 0$ such that when $n \geq N_\gamma$,

$$\nu\left(\{x \in \Sigma_A \colon d_\theta(x,L_n) \geq \theta^{m_n}\}\right) > \gamma$$

for every invariant measure $\nu \in \mathcal{M}_\sigma$ such that $h(\nu) \geq 2\gamma h_{top}(\sigma)$.

*Proof of claim.* For each $k > 0$, let $\Omega_k$ be the set of all closed $\theta^k$-balls (or equivalently $k$-cylinders) in $\Sigma_A$. Note that for each $k$, $\Omega_k$ is a finite partition of $\Sigma_A$ into clopen sets. Recall that

$$h_{top}(\sigma) = \lim_{k\to\infty}\frac{1}{k}\log\#\Omega_k.$$

Since $m_n^{-1}\log n \to 0$ by the claim in Step 3, there exists $N_\gamma$ such that if $n \geq N_\gamma$, then

$$\frac{\log n}{m_n} + \gamma\frac{\log\#\Omega_{m_n}}{m_n} + \frac{2}{m_n} < 2\gamma h_{top}(\sigma). \tag{4}$$

Now let $\nu \in \mathcal{M}_\sigma$ and suppose that

$$\nu\left(\{x \in \Sigma_A \colon d_\theta(x,L_n) \geq \theta^{m_n}\}\right) \leq \gamma \tag{5}$$

for some $n \geq N_\gamma$. We will show that necessarily $h(\nu) < 2\gamma h_{top}(\sigma)$, which proves the claim.

Recall from e.g. [18] that

$$h(\nu) = \inf_{k\geq 1}\frac{1}{k}\sum_{\omega\in\Omega_k} -\nu(\omega)\log\nu(\omega).$$

Let $W_n \subseteq \Omega_{m_n}$ be the set of all elements of $\Omega_{m_n}$ which intersect $L_n$. Since $\mu_n$ is a periodic orbit of period not more than $n$, it follows that $W_n$ has cardinality at most $n$. Define

$$\tilde{\gamma} := \sum_{\omega\in\Omega_{m_n}\setminus W_n}\nu(\omega) = \nu\left(\{x \in \Sigma_A \colon d_\theta(x,L_n) > \theta^{m_n}\}\right).$$

If (5) holds then clearly $0 \leq \tilde{\gamma} \leq \gamma$. Using Lemma 2.1 in combination with (4), it follows that

$$\begin{aligned} h(\nu) &\leq \frac{1}{m_n}\sum_{\omega\in W_n} -\nu(\omega)\log\nu(\omega) + \frac{1}{m_n}\sum_{\omega\in\Omega_{m_n}\setminus W_n} -\nu(\omega)\log\nu(\omega) \\ &\leq \frac{1-\tilde{\gamma}}{m_n}\log n + \frac{\tilde{\gamma}}{m_n}\log\#\Omega_{m_n} + \frac{2}{m_n} < 2\gamma h_{top}(\sigma), \end{aligned}$$

which proves the claim.

**Step 5.** We now complete the proof. Define a sequence of functions $f_n \in H_\theta$ by

$$f_n(x) = f(x) + \varepsilon - \varepsilon d_\theta(x,L_n), \tag{6}$$

where $L_n = \operatorname{supp}\mu_n$ as above. From the definition of $\varepsilon$ we have $f_n \in U$ for each $n \geq 1$. From Step 3 we have

$$\int d_\theta(x, K)\, d\mu_n = o(\theta^{m_n}),$$

and from Step 4 it follows that when $n$ is sufficiently large,

$$\int d_\theta(x, L_n)\, d\nu \geq \theta^{m_n}\nu\left(\{x \in \Sigma_A \colon d_\theta(x, L_n) \geq \theta^{m_n}\}\right) \geq \gamma\theta^{m_n}$$

for all $\nu \in \mathcal{M}_\sigma$ such that $h(\nu) \geq 2\gamma h_{top}(\sigma)$.

We may therefore choose $n$ such that $\varepsilon \int d_\theta(x, L_n)\, d\nu > C \int d_\theta(x, K)\, d\mu_n$ for every $\nu \in \mathcal{M}_\sigma$ such that $h(\nu) \geq 2\gamma h_{top}(\sigma)$. It follows that for every such measure $\nu$

$$\begin{aligned}\int f_n\, d\nu &= \int f\, d\nu + \varepsilon - \varepsilon \int d_\theta(x, L_n)\, d\nu \\ &< \beta(f) - C\int d_\theta(x, K)\, d\mu_n + \varepsilon \\ &\leq \int f\, d\mu_n + \varepsilon = \int f_n\, d\mu_n \leq \beta(f_n),\end{aligned}$$

where we have applied (3) and (6). We have shown that if $\nu \in \mathcal{M}_\sigma$ and $h(\nu) \geq 2\gamma h_{top}(\sigma)$, then $\nu \notin \mathcal{M}_{\max}(f_n)$, and therefore $f_n \in \mathcal{E}_\gamma \cap U$. We conclude that $\mathcal{E}_\gamma$ is dense in $H_\theta$ and the theorem is proved.

## 4. Proof of Theorem 2

The proof of Theorem 2 follows the same general lines as that of Theorem 1. In this section we briefly outline the similarities and differences. Let $T \colon S^1 \to S^1$ be a $C^2$ expanding map of degree $D$, $|D| \geq 2$. As in Theorem 1, we begin by defining

$$\mathcal{E}_\gamma = \left\{f \in C_{Lip}(S^1) \colon h(\mu) < 2\gamma \log|D| \text{ for every } \mu \in \mathcal{M}_{\max}(f)\right\}$$

for each $\gamma > 0$, and proceed to show that each $\mathcal{E}_\gamma$ is open and dense in $C_{Lip}(S^1)$. **Step 1** proceeds by direct analogy with Theorem 1. We require an additional step:

**Step 1A.** Let $P$ be a partition of $S^1$ into $|D|$ intervals such that $TI = S^1$ for each $I \in P$, and

$$\bigvee_{j=0}^{\infty} T^{-j}P = \mathcal{B} \tag{7}$$

where $\mathcal{B}$ is the Borel $\sigma$-algebra of $S^1$. The equation (7) implies that for every $\nu \in \mathcal{M}_T$

$$h(\nu) = \inf_{k\geq 1} \frac{1}{k} \sum_{I \in \bigvee_{j=0}^{k-1} T^{-j}P} -\nu(I)\log\nu(I),$$

see e.g. [18].

We make the following claim: there exists $\tau > 0$ such that every ball of radius $\tau^n$ in $S^1$ intersects no more than 3 elements of $\bigvee_{j=0}^{n-1} T^{-j}P$. To prove the claim, it suffices to show that we may choose $\tau$ small enough that every $I \in \bigvee_{j=0}^{n-1} T^{-j}P$ is of length not less than $\tau^n$.

In **Step 2** we proceed as for Theorem 1. In **Step 3** we proceed as before, except that we choose the sequences $(\mu_n)_{n\geq 1}$ and $(m_n)_{n\geq 1}$ such that

$$\int d(x,K)\,d\mu_n = o(\tau^{m_n})$$

and $m_n^{-1}\log n \to 0$. In this case the relevant result in the article of Bressaud and Quas is [6, Theorem 3].

**Step 4** takes the following form: we wish to choose $N_\gamma > 0$ such that when $n \geq N_\gamma$,

$$\nu\left(\left\{x \in S^1 \colon d(x,L_n) \geq \tau^{m_n}\right\}\right) > \gamma$$

for every invariant measure $\nu \in \mathcal{M}_T$ such that $h(\nu) \geq 2\gamma \log|D|$. To see that this is possible, choose $N_\gamma$ large enough that $n \geq N_\gamma$ implies

$$\frac{\log n}{m_n} + \gamma \log|D| + \frac{2+\log 3}{m_n} < 2\gamma \log|D|.$$

Let $n \geq N_\gamma$ and $\nu \in \mathcal{M}_T$, and suppose that

$$\nu\left(\left\{x \in S^1 \colon d(x,L_n) \geq \tau^{m_n}\right\}\right) \leq \gamma. \tag{8}$$

Let $W_n$ be the set of all intervals $J \in \bigvee_{j=0}^{m_n-1} T^{-j}P$ such that $d(x,L_n) < \tau^{m_n}$ for some $x \in J$. Note that

$$\bigcup_{I \in \left(\bigvee_{j=0}^{m_n-1} T^{-j}P\right)\setminus W_n} I \subseteq \left\{x \in S^1 \colon d(x,L_n) \geq \tau^{m_n}\right\}. \tag{9}$$

The claim in Step 1A implies that the cardinality of $W_n$ is at most $3n$, and if we define

$$\tilde{\gamma} := \sum_{I \in \left(\bigvee_{j=0}^{m_n-1} T^{-j}P\right)\setminus W_n} \nu(I)$$

then clearly (8) and (9) yield $0 \leq \tilde{\gamma} \leq \gamma$. We deduce that

$$h(\nu) \leq \frac{1-\tilde{\gamma}}{m_n}\log n + \tilde{\gamma}\log|D| + \frac{2+\log 3}{m_n} < 2\gamma\log|D|$$

as in the previous section, which completes the proof of the claim. **Step 5** now proceeds after the same fashion and Theorem 2 is proved.

## References


[1] T. Bousch. Le poisson n'a pas d'arêtes. *Ann. Inst. H. Poincaré Proba. Statist.* **36** (2000) 489-508.

[2] T. Bousch. La condition de Walters. *Ann. Sci. École Norm. Sup.* **34** (2001) 287-311.

[3] T. Bousch. Nouveulle preuve d'un théorème de Yuan et Hunt. Preprint November 2006.

[4] T. Bousch and J. Mairesse. Asymptotic height optimisation for topical IFS, Tetris heaps, and the finiteness conjecture. *J. Amer. Math. Soc.* **15** (2002) 77-111.

[5] J. Brémont. Gibbs measures at temperature zero. *Nonlinearity* (2003) **16** 419-426.

[6] X. Bressaud and A. Quas. Rates of approximation of minimizing measures. *Nonlinearity* **20** (2007) 845-853.

[7] G. Contreras, A. O. Lopes and P. Thieullen. Lyapunov minimizing measures for expanding maps of the circle. *Ergodic Theory Dyn. Systems* **21** (2001).

[8] J.-P. Conze and Y. Guivarc'h. Croissance des sommes ergodiques. Unpublished manuscript (*c.* 1993).

[9] M. Denker, C. Grillenberger and K. Sigmund. *Ergodic theory on compact spaces.* Lecture Notes in Mathematics 527. Springer-Verlag, Berlin-New York (1976).

[10] B. R. Hunt and E. Ott. Optimal periodic orbits of chaotic systems.*Phys. Rev. Lett.* **76** (1996) 2254-2257.

[11] O. Jenkinson. Frequency locking on the boundary of the barycentre set. *Experimental Mathematics* **9** (2000) 309-317.
[12] O. Jenkinson. Ergodic optimization. *Discrete Contin. Dyn. Syst.* **15** (2006) 197-224.
[13] O. Jenkinson, R. D. Mauldin and M. Urbański. Zero-temperature limits of Gibbs-equilibrium states for countable alphabet subshifts of finite type. *J. Statist. Phys.* **119** (2005) 765-776.
[14] N. N. Nekhoroshev. Asymptotics of Gibbs measures in one-dimensional lattice models. *Moscow Univ. Math. Bull.* **59** (2004) 10-15.
[15] W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics.* Astérique **187-188** (1990).
[16] D. Ruelle. *Thermodynamic formalism.* Encyclopedia of Mathematics and its Applications, **5** (1978).
[17] S. V. Savchenko. Homological inequalities for finite topological Markov chains. *Funct. Anal. Appl.* **33** (1999) 236-238.
[18] P. Walters. *An Introduction to ergodic theory.* Graduate texts in Mathematics 79. Springer-Verlag, New York-Berlin, 1982.
[19] G. Yuan and B. R. Hunt. Optimal orbits of hyperbolic systems. *Nonlinearity* **12** (1999) 1207-1224.

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