RIGHT ORDERABILITY AND GRAPHS OF GROUPS

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1. Introduction

In [1, Theorem A], a criterion for a free product with amalgamation to be right orderable is given, in terms of families of right orders on the free factors. There is a corresponding result for an HNN-extension with a single pair of associated subgroups, and these lead to many interesting consequences. The purpose of this note is to give a generalisation to the fundamental group of any graph of groups.

Recall that a group is called *tree-free* if it has an action on a Λ-tree, for some ordered abelian group Λ, which is free and without inversions. At the end of Chapter 5 in [3] it is asked whether or not all tree-free groups are orderable, or at least right-orderable. The context is that groups in the class of locally fully residually free groups, a large class of tree-free groups, are (two-sided) orderable. It was shown in [4] that tree-free groups are unique product groups. Using results of Guirardel ([6]), the main result here can be used to prove that, if $\Lambda = \mathbb{R}^n$, $n \geq 1$, with the lexicographic order, then groups having a free action without inversions on a Λ-tree are right orderable.

The notation used for graphs of groups is that in [4] [8, §§5.1 and 5.3, Ch.I], but with some changes. The vertices and edges of a graph Y will be denoted by $V(Y)$, $E(Y)$ respectively. Also in Definition (b) of fundamental group in [8, §5.1, Ch.I], the generator g_y will be denoted by q_y (and e, f etc, rather than y, will be used for edges.) If (G,Y) is a graph of groups and T is a maximal tree of Y, $\pi(G,Y,T)$ will be used instead of $\pi_1(G,Y,T)$. Thus $\pi(G,Y,T)$ is the group with presentation

$$\langle q_e\ (e \in E(Y)),\ G_v\ (v \in V(Y)) \mid \mathrm{rel}(G_v),\ q_e a^e q_e^{-1} = a^{\bar{e}}\ (a \in G_e, e \in E(Y)),$$
$$q_e q_{\bar{e}} = 1\ (e \in E(Y)), q_e = 1\ (e \in E(T))\rangle$$

where $a \mapsto a^e$, $G_e \to G_{t(e)}$ is the monomorphism given by the graph of groups. Finally, the tree constructed in [8, §5.3, Ch.I] is written as $\widetilde{Y}(G,Y,T)$ rather than $\widetilde{X}(G,Y,T)$.

If (G,Y) is a graph of groups and Y' is a connected subgraph of Y, the restriction of (G,Y) to Y' is denoted by $(G|_{Y'},Y')$. If T' is a maximal tree of Y' and T is an extension of T' to a maximal tree of Y, then there is a canonical inclusion $\pi(G|_{Y'},Y',T') \to \pi(G,Y,T)$ (see [5, Lemma 19, Ch. 8]). This applies in particular when Y' consists of a single vertex, so the vertex groups embed canonically in $\pi(G,Y,T)$, and will be identified with their images in $\pi(G,Y,T)$.

To conform with [8], mappings will be written on the left, and group actions on trees will be left actions, although this is not the best convention when dealing with right orders.

If G is a group acting on a set X, then for $x \in X$, $\mathrm{stab}(x)$, or $\mathrm{stab}_G(x)$, if necessary, denotes the stabilizer of x in G.

Some terminology concerning right orders, mostly taken from [1], but with some minor changes, needs to be established. If $\varphi : H \to G$ is a monomorphism of groups, and $\leq$ is a right order on G, there is a right order $\leq'$ on H defined by: $h_1 \leq' h_2$ if and only if $\varphi(h_1) \leq \varphi(h_2)$. This is called the order induced by $\leq$ on H (via φ). This will mainly be used when H is a subgroup of G and φ is the inclusion map, in which case $\leq$ is an extension of $\leq'$.

If $\leq$ is a right order on a group H_1, and $\varphi : H_1 \to H_2$ is an isomorphism of groups, then there is an induced right order on H_2 via φ^{-1}, which is denoted by $\leq^{\varphi}$. Thus $x \leq^{\varphi} y \iff \varphi^{-1}(x) \leq \varphi^{-1}(y)$. The orders $\leq$, $\leq^{\varphi}$, are called *corresponding orders under* φ.

A special case is when H_1 is a subgroup of a group G, $g \in G$ and $H_2 = g^{-1}H_1g$, and φ is defined by: $\varphi(x) = g^{-1}xg$. In this case the order $\leq^{\varphi}$ is denoted by $\leq^{g}$.

Definition. A set $\mathcal{R}$ of right orders on a group G is called *normal* (or *G-invariant*) if it is non-empty and, for all $g \in G$ and $\leq$ belonging to $\mathcal{R}$, $\leq^{g}$ belongs to $\mathcal{R}$.

Definition. Let G_1, G_2 be groups, let H_i be a subgroup of G_i and let $\leq_i$ be a right order on G_i, for $i = 1,\ 2$. Suppose $\varphi : H_1 \to H_2$ is an isomorphism. Then φ is *compatible for* $(\leq_1, \leq_2)$ if, for all $h \in H_1$, $1 \leq_1 h$ implies $1 \leq_2 \varphi(h)$.

If $\mathcal{R}_i$ is a set of right orders on G_i, for $i = 1,2$, then φ is *compatible for* $(\mathcal{R}_1, \mathcal{R}_2)$ if for all $\leq_1$ in $\mathcal{R}_1$, there exists $\leq_2$ in $\mathcal{R}_2$ such that φ is compatible for $(\leq_1, \leq_2)$.

If φ is compatible for $(\leq_1, \leq_2)$, then since these are linear orders, it follows that, for all $h \in H_1$, $1 \leq_1 h$ if and only if $1 \leq_2 \varphi(h)$; that is, the orders induced by $\leq_i$ on H_i are corresponding orders under the isomorphism φ. Also, φ^{-1} is compatible for $(\leq_2, \leq_1)$.

Also, the ideas of *ultraproduct of orders* and of *D-order* will be used, and the reader is referred to [1] for the definition. Again from [1], a set $\mathcal{R}$ of right orders on a group G is called $\mathfrak{U}$*-closed* if it is closed under ultraproducts of orders from $\mathcal{R}$, and *D-invariant* if it is closed under taking D-orders determined by orders in $\mathcal{R}$. Finally, $\mathcal{R}$ is called *$\mathcal{A}$-invariant* if it is normal, $\mathfrak{U}$-closed and D-invariant.

2. The Case of a Tree of Groups

The criterion for the fundamental group of a tree of groups to be right orderable uses the following idea of compatibility, which applies to any graph of groups.

Definition. Let (G,Y) be a graph of groups, and let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups, for $e \in E(Y)$ (in the notation of [8], $\varphi_e(a) = a^e$). For each $v \in V(Y)$, let $\leq_v$ be a right order on G_v. The family $\{\leq_v \mid v \in V(Y)\}$ is said to be *compatible for* (G,Y) if $\varphi_e\varphi_{\bar{e}}^{-1}$ $(a^{\bar{e}} \mapsto a^e)$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(Y)$.

More generally, let $\mathcal{R}_v$ be a set of right orders on G_v. The family $\{\mathcal{R}_v \mid v \in V(Y)\}$ is said to be *compatible for* (G,Y) if $\varphi_e\varphi_{\bar{e}}^{-1}$ is compatible for $(\mathcal{R}_{o(e)}, \mathcal{R}_{t(e)})$, for all $e \in E(Y)$.

Definition. Let (G,Y) be a graph of groups, and, for each $v \in V(Y)$, let $\mathcal{R}_v$ be a set of right orders on G_v. The family $\{\mathcal{R}_v \mid v \in V(Y)\}$ is called *normal* if every $\mathcal{R}_v$ is a normal set of right orders on G_v.

The next step is to note that Lemma 4.3 in [1] easily generalises to give the following lemma.

Lemma 2.1. *Let $\{\mathcal{R}_v \mid v \in V(Y)\}$ be a normal, compatible family of sets of right orders for a graph of groups (G,Y). Then there is a compatible family of sets of right orders for (G,Y), $\{\overline{\mathcal{R}}_v \mid v \in V(Y)\}$, such that $\mathcal{R}_v \subseteq \overline{\mathcal{R}}_v$ and $\overline{\mathcal{R}}_v$ is $\mathcal{A}$-invariant, for all $v \in V(Y)$.*

Proof. Let $\{\{\mathcal{R}_{v,i} \mid v \in V(Y)\} \mid i \in I\}$ be the collection of all families of normal, compatible sets of right orders for (G,Y), indexed by some set I. Put $\overline{\mathcal{R}}_v := \bigcup_{i\in I} \mathcal{R}_{v,i}$. Then $\{\overline{\mathcal{R}}_v \mid v \in V(Y)\}$ is plainly a normal, compatible family of sets of right orders for (G,Y), with $\mathcal{R}_v \subseteq \overline{\mathcal{R}}_v$.

Suppose $\mathcal{R}_v^U$ is the set of all ultraproducts of orders from $\overline{\mathcal{R}}_v$, for $v \in V(Y)$. From the proof of [1, Lemma 4.3], $\{\mathcal{R}_v^U \mid v \in V(Y)\}$ is a normal, compatible family for (G,Y), hence $\mathcal{R}_v^U \subseteq \overline{\mathcal{R}}_v$, for all $v \in V(Y)$. Therefore $\overline{\mathcal{R}}_v$ is $\mathfrak{U}$-closed.

Now let $\mathcal{R}_v^D$ be the set of all D-orders arising from orders in $\overline{\mathcal{R}}_v$, for $v \in V(Y)$. From the proof of [1, Lemma 4.3], $\{\mathcal{R}_v^D \mid v \in V(Y)\}$ is a normal, compatible family for (G,Y), hence $\mathcal{R}_v^D \subseteq \overline{\mathcal{R}}_v$, for all $v \in V(Y)$. Therefore $\overline{\mathcal{R}}_v$ is D-invariant. Thus $\overline{\mathcal{R}}_v$ is $\mathcal{A}$-invariant, for all $v \in V(Y)$, and $\{\overline{\mathcal{R}}_v \mid v \in V(Y)\}$ is the required family. □

The criterion for a tree of groups can now be proved, starting with a finite tree, then using this to obtain the general result. The next lemma is the main point at which input from [1] is needed; the proof of Theorem A given there establishes the lemma in the case that Y has one unoriented edge, and permits an inductive argument.

Lemma 2.2. *Suppose Y is a finite tree, (G,Y) is a graph of groups and $\{\mathcal{R}_v \mid v \in V(Y)\}$ is a normal, compatible family of sets of right orders for (G,Y). Then $\pi(G,Y,Y)$ is right orderable.*

If $\leq_v$ belongs to $\mathcal{R}_v$, for all $v \in V(Y)$, and $\{\leq_v \mid v \in V(Y)\}$ is compatible for (G,Y), then there is a right order $\leq$ on $\pi(G,Y,Y)$ such that $\leq$ induces $\leq_v$ on G_v, for all $v \in V(Y)$.

Proof. Let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups, for $e \in E(Y)$, and let $\pi = \pi(G,Y,Y)$. Assuming all $\mathcal{R}_v$ are $\mathcal{A}$-invariant, it will be shown that there is an $\mathcal{A}$-invariant set $\mathcal{R}$ of right orders on π satisfying

(1) the set of orders induced on G_v by the orders in $\mathcal{R}$ is $\mathcal{R}_v$, for all $v \in V(Y)$;
(2) if $\leq_v$ belongs to $\mathcal{R}_v$, for all $v \in V(Y)$, and $\{\leq_v \mid v \in V(Y)\}$ is compatible for (G,Y), then there is a right order $\leq$ in $\mathcal{R}$ such that $\leq$ induces $\leq_v$ on G_v, for all $v \in V(Y)$.

By Lemma 2.1, this is enough to prove the lemma. The proof is by induction on the number of vertices of Y. If Y has one vertex, there is nothing to prove, otherwise let z be a terminal vertex of Y, $e, \bar{e}$ the edges incident with z, with $t(e) = z$, and Y' the subtree obtained by removing z, e and $\bar{e}$. Then (see [8, Example (b), §4.4, Ch. I])

$$\pi = G_z *_{G_e} \pi(G|_{Y'}, Y', Y')$$

and Y' has one less vertex than Y. (The free product is formed using the monomorphisms φ_e, and $\varphi_{\bar{e}}$ composed with the inclusion map $G_{o(e)} \to \pi(G|_{Y'}, Y', Y')$.) By induction, there is an $\mathcal{A}$-invariant set of right orders, $\mathcal{R}'$, on $\pi(G|_{Y'}, Y', Y')$ satisfying

(1)′ the set of orders induced on G_v by the orders in $\mathcal{R}'$ is $\mathcal{R}_v$, for all $v \in V(Y')$;
(2)′ if $\leq_v$ belongs to $\mathcal{R}_v$, for all $v \in V(Y')$, and $\{\leq_v | \, v \in V(Y')\}$ is compatible for $(G|'_Y, Y')$, then there is a right order $\leq'$ in $\mathcal{R}'$ such that $\leq'$ induces $\leq_v$ on G_v, for all $v \in V(Y')$.

In particular, the set of orders induced on $G_{o(e)}$ by orders in $\mathcal{R}'$ is $\mathcal{R}_{o(e)}$. It follows that $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\mathcal{R}', \mathcal{R}_v)$ and $\varphi_{\bar{e}} \varphi_e^{-1}$ is compatible for $(\mathcal{R}_v, \mathcal{R}')$. Further, if $\{\leq_v | \, v \in V(Y)\}$ is as in (2), then (2)′ applies to $\{\leq_v | \, v \in V(Y')\}$, to give an order $\leq'$ in $\mathcal{R}'$ such that $\leq'$ induces $\leq_v$ on G_v, for all $v \in V(Y')$.

From the proof of Theorem A in [1], there is an $\mathcal{A}$-invariant set $\mathcal{R}$ of orders on π such that

(a) the set of orders induced on G_z by the orders in $\mathcal{R}$ is $\mathcal{R}_z$;
(b) the set of orders induced on $\pi(G|_{Y'}, Y', Y')$ by the orders in $\mathcal{R}$ is $\mathcal{R}'$;
(c) if $\{\leq_v | \, v \in V(Y)\}$ is as in (2), then there is a right order $\leq$ in $\mathcal{R}$ such that $\leq$ induces $\leq_z$ on G_z, and $\leq'$ on $\mathcal{R}'$.

It follows easily that $\mathcal{R}$ satisfies (1) and (2), completing the induction. □

Lemma 2.3. *Suppose (G,Y) is a graph of groups, where Y is a tree, and $\{\mathcal{R}_v \mid v \in V(Y)\}$ is a normal, compatible family of right orders for (G,Y). Then $\pi(G,Y,Y)$ is right orderable.*

Let $\leq_v$ belong to $\mathcal{R}_v$, for all $v \in V(Y)$ and assume $\{\leq_v | \, v \in V(Y)\}$ is compatible for (G,Y). Then there is a right order $\leq$ on $\pi(G,Y,Y)$, such that $\leq$ induces $\leq_v$ on G_v, for all $v \in V(Y)$.

Proof. Let $\pi = \pi(G,Y,Y)$; then π is generated by $\bigcup_{v \in V(Y)} G_v$. Hence, if H is a finitely generated subgroup of π, there is a finite subtree Y' of Y such that H is a subgroup of $\pi(G|_{Y'}, Y', Y')$. By Lemma 2.2, $\pi(G|_{Y'}, Y', Y')$ is right orderable, hence so is H. It follows that π is right orderable.

For the second part, let $\mathcal{F}$ be the set of all finite subtrees of Y. For $Z \in \mathcal{F}$, let

$$a_Z = \left\{ Z' \in \mathcal{F} \mid Z \text{ is a subtree of } Z' \right\}.$$

Then $a_Z \neq \emptyset$ as $Z \in a_Z$. Also, if $Z, Z' \in \mathcal{F}$, then there is a finite subtree Z'' of Y containing both Z and Z', so $a_Z \cap a_{Z'} \supseteq a_{Z''}$. Hence there is an ultrafilter $\mathcal{U}$ on $\mathcal{F}$ such that $a_Z \in \mathcal{U}$ for all $Z \in \mathcal{F}$.

Using Lemma 2.2, for each $Z \in \mathcal{F}$ choose a right order $\leq_Z$ on $\pi(G|_Z, Z, Z)$ such that, for all $v \in V(Z)$, $\leq_Z$ induces $\leq_v$ on G_v.

Let $\tilde{\pi} = \prod_{Z \in \mathcal{F}} \pi(G|_Z, Z, Z)/\mathcal{U}$, an ultraproduct of right ordered groups, so a right ordered group via $\leq_{\mathcal{U}}$, the ultraproduct of the orders $\leq_Z$, for $Z \in \mathcal{F}$.

Define $\phi : \pi \longrightarrow \tilde{\pi}$ by $\phi(g) = [\phi_Z(g)]$, the equivalence class of $(\phi_Z(g))_{Z\in\mathcal{F}}$ in $\tilde{\pi}$, where

$$\phi_Z(g) = \begin{cases} g & \text{if } g \in \pi(G|_Z, Z, Z) \\ 1 & \text{otherwise.} \end{cases}$$

If $x, y \in \pi$, choose $Z_0 \in \mathcal{F}$ such that $x, y \in \pi(G|_{Z_0}, Z_0, Z_0)$. Then

$$\phi_Z(xy) = xy = \phi_Z(x)\phi_Z(y)$$

for all $Z \in a_{Z_0}$. Since $a_{Z_0} \in \mathcal{U}$, $\phi(xy) = \phi(x)\phi(y)$, so ϕ is a group homomorphism.

Suppose $\phi(x) = 1$; then $\phi_Z(x) = 1$ for all $Z \in A$, where A is some element of $\mathcal{U}$. Choose $Z_0 \in \mathcal{F}$ such that $x \in \pi(G|_{Z_0}, Z_0, Z_0)$. Then $a_{Z_0} \cap A \in \mathcal{U}$, so is non-empty. Take $Z \in a_{Z_0} \cap A$; then $\phi_Z(x) = 1$, and $x \in \pi(G|_Z, Z, Z)$, so $\phi_Z(x) = x$. Hence $x = 1$ and so ϕ is an embedding, and $\leq_{\mathcal{U}}$ induces a right order on π via ϕ, denoted by $\leq$.

Let $v \in V(Y)$, and let $x, y \in G_v$. Choose $Z_0 \in \mathcal{F}$ such that $v \in V(Z_0)$. Suppose $x \leq_v y$; then for all $Z \in a_{Z_0}$,

$$\phi_Z(x) = x \leq_Z y = \phi_Z(y)$$

and $a_{Z_0} \in \mathcal{U}$, so $\phi(x) \leq_{\mathcal{U}} \phi(y)$, that is, $x \leq y$.

Suppose $x \leq y$, so $\phi(x) \leq_{\mathcal{U}} \phi(y)$, hence there exists $A \in \mathcal{U}$ such that $\phi_Z(x) \leq_Z \phi_Z(y)$ for all $Z \in A$. Since $a_{Z_0} \cap A \in \mathcal{U}$, there exists $Z \in a_{Z_0} \cap A$. Then Z_0 is a subtree of Z, so x, $y \in \pi(G|_Z, Z, Z)$, and $x = \phi_Z(x) \leq_Z \phi_Z(y) = y$. Since $v \in V(Z)$, the order $\leq_Z$ induces $\leq_v$ on G_v, hence $x \leq_v y$. Thus $\leq$ induces $\leq_v$ on G_v, as required. □

3. The General Case and Applications

Assume (G,Y) is a graph of groups and $\{\mathcal{R}_v \mid v \in V(Y)\}$ is a normal, compatible family of sets of right orders for (G,Y). Let T be a maximal tree of Y and put $\pi = \pi(G,Y,T)$. Choose an orientation A of Y and let $\widetilde{Y} = \widetilde{Y}(G,Y,T)$ be the canonical tree on which π acts, formed using the orientation A. Recall ([8, §5.4, Ch. I]) that $V(\widetilde{Y}) = \coprod_{v\in V(Y)} \pi/G_v$, where $\pi/G_v = \{gG_v \mid g \in \pi\}$, and the action of π on $V(\widetilde{Y})$ is given by the usual action on cosets of a subgroup. For $v \in V(Y)$, $\tilde{v}$ means the vertex corresponding to G_v in this disjoint union, so $\mathrm{stab}(\tilde{v}) = G_v$. For $e \in E(Y)$, G_e^e denotes the image of G_e under the monomorphism $G_e \to G_{t(e)}$ given by the graph of groups, and $E(\widetilde{Y}) = \coprod_{e\in E(Y)} \pi/G'_e$, where

$$G'_e = \begin{cases} G_{\bar{e}}^{\bar{e}} & \text{if } e \in A \\ G_e^e & \text{if } e \notin A \end{cases}$$

For $e \in E(Y)$, $\tilde{e}$ is defined to be the edge corresponding to the coset G'_e in the disjoint union (care is needed as $G'_e = G'_{\bar{e}}$).

Let $u \in V(\widetilde{Y})$, so $u = g\tilde{v}$ for some $g \in \pi$ and $v \in V(Y)$. For $\leq$ in $\mathcal{R}_v$, the right order $\leq^{g^{-1}}$ on $\mathrm{stab}(u) = gG_vg^{-1}$ is given by

$$a \leq^{g^{-1}} b \Longleftrightarrow g^{-1}ag \leq g^{-1}bg \quad (a,\, b \in \mathrm{stab}(u)).$$

The set $\mathcal{R}_{v,g} := \{\leq^{g^{-1}} | \leq$ belongs to $\mathcal{R}_v\}$ is independent of the choice of g (in the coset gG_v). To see this, suppose $g_1 = gh$, where $h \in G_v$, and $\leq$ is equal to $\leq_1^{g^{-1}}$, where $\leq_1$ is in $\mathcal{R}_v$. Then, for $a, b \in \text{stab}(u)$,

$$\begin{aligned} a \leq b &\Longleftrightarrow g^{-1}ag \leq_1 g^{-1}bg \\ &\Longleftrightarrow hg_1^{-1}ag_1h^{-1} \leq_1 hg_1^{-1}bg_1h^{-1} \\ &\Longleftrightarrow g_1^{-1}ag_1 \leq_2 g_1^{-1}bg_1 \end{aligned}$$

where $\leq_2$ is $\leq_1^h$, which is in $\mathcal{R}_v$, because $\mathcal{R}_v$ is normal. Thus $\leq$ is equal to $\leq_2^{g_1^{-1}}$, which belongs to $\mathcal{R}_{v,g_1}$, and so $\mathcal{R}_{v,g} \subseteq \mathcal{R}_{v,g_1}$. By symmetry, since $g = g_1h^{-1}$, $\mathcal{R}_{v,g} = \mathcal{R}_{v,g_1}$, as claimed.

Thus it is legitimate to define $\mathcal{R}_u := \mathcal{R}_{v,g}$.

Lemma 3.1. *In this situation, the set $\mathcal{R}_u$ is a normal set of right orders on stab(u), for all $u \in V(\widetilde{Y})$.*

Proof. Let $u \in V(\widetilde{Y})$, so $u = g\tilde{v}$ for some $v \in V(Y)$ and $g \in \pi$. Suppose $h \in \text{stab}(u) = gG_vg^{-1}$, so $h = gkg^{-1}$ for some $k \in G_v$, and suppose $\leq$ is in $\mathcal{R}_u$. Then $\leq$ is $\leq_1^{g^{-1}}$ for some $\leq_1$ in $\mathcal{R}_v$. For $a, b \in \text{stab}(u)$,

$$\begin{aligned} a \leq^h b &\Longleftrightarrow ha \leq hb \\ &\Longleftrightarrow g^{-1}hag \leq_1 g^{-1}hbg \\ &\Longleftrightarrow kg^{-1}ag \leq_1 kg^{-1}bg \\ &\Longleftrightarrow g^{-1}ag \leq_2 g^{-1}bg \end{aligned}$$

where $\leq_2$ is $\leq_1^k$, which is in $\mathcal{R}_v$ as this is a normal set of right orders on G_v. Thus $\leq^h$ is $\leq_2^{g^{-1}}$, which is in $\mathcal{R}_u$, hence $\mathcal{R}_u$ is normal. □

Lemma 3.2. *In the situation of Lemma 3.1, the identity map stab$(f) \to$ stab(f) is compatible for $(\mathcal{R}_{o(f)}, \mathcal{R}_{t(f)})$, for all $f \in E(\widetilde{Y})$.*

Proof. Write $f = g\tilde{e}$, where $g \in \pi$ and $e \in E(Y)$. There are two similar cases to consider.
Case 1: $e \in A$. Then (cf [8, §5.3, Ch. I])

$$\begin{aligned} t(f) &= gq_e\widetilde{t(e)} \\ o(f) &= g\widetilde{o(e)} \\ \text{stab}(f) &= gG_{\bar{e}}^{\bar{e}}g^{-1}. \end{aligned}$$

Suppose $\leq$ is in $\mathcal{R}_{o(f)}$. By definition, there is $\leq_1$ in $\mathcal{R}_{o(e)}$ such that, for $a, b \in \text{stab}(o(f)) = gG_{o(e)}g^{-1}$,

$$a \leq b \Longleftrightarrow g^{-1}ag \leq_1 g^{-1}bg.$$

An element of $\mathrm{stab}(f)$ has the form $gc^{\bar{e}}g^{-1}$, where $c \in G_e$. For such c,

$$1 \leq gc^{\bar{e}}g^{-1} \Longleftrightarrow 1 \leq_1 c^{\bar{e}}$$

and, by assumption, there exists $\leq_2$ in $\mathcal{R}_{t(e)}$ such that

$$1 \leq_1 c^{\bar{e}} \Longleftrightarrow 1 \leq_2 c^{e}.$$

Let $\leq_3$ be the right order $\leq_2^{(gq_e)^{-1}}$ in $\mathcal{R}_{t(f)}$. Then $1 \leq_2 c^e$ implies $1 \leq_3 gq_ec^eq_e^{-1}g^{-1} = gc^{\bar{e}}g^{-1}$. Thus the identity map on $\mathrm{stab}(f)$ is compatible for $(\leq, \leq_3)$.

Case 2: $e \notin A$. Then

$$t(f) = g\widetilde{t(e)}$$
$$o(f) = gq_e^{-1}\widetilde{o(e)}$$
$$\mathrm{stab}(f) = gG_e^eg^{-1}.$$

Suppose $\leq$ is in $\mathcal{R}_{o(f)}$, so there exists $\leq_1$ in $\mathcal{R}_{o(e)}$ such that, for a, $b \in \mathrm{stab}(o(f)) = gq_e^{-1}G_{o(e)}q_eg^{-1}$,

$$a \leq b \Longleftrightarrow q_eg^{-1}agq_e^{-1} \leq_1 q_eg^{-1}bgq_e^{-1}.$$

An element of $\mathrm{stab}(f)$ has the form gc^eg^{-1}, where $c \in G_e$. For such c,

$$1 \leq gc^eg^{-1} \Longleftrightarrow 1 \leq_1 q_ec^eq_e^{-1} = c^{\bar{e}}$$

and, by assumption, there exists $\leq_2$ in $\mathcal{R}_{t(e)}$ such that

$$1 \leq_1 c^{\bar{e}} \Longleftrightarrow 1 \leq_2 c^{e}.$$

Let $\leq_3$ be the right order $\leq_2^{g^{-1}}$ in $\mathcal{R}_{t(f)}$. Then $1 \leq_2 c^e$ implies $1 \leq_3 gc^eg^{-1}$. Thus the identity map on $\mathrm{stab}(f)$ is compatible for $(\leq, \leq_3)$. The lemma follows. □

The main result can now be proved.

Theorem 3.3. *Let (G,Y) be a graph of groups, let T be a maximal tree of Y and let $\pi = \pi(G,Y,T)$. Then π is right orderable if and only if there is a normal, compatible family of sets of right orders $\{\mathcal{R}_v \mid v \in V(Y)\}$ for (G,Y).*

Let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups, for $e \in E(Y)$. Suppose $\leq_v$ is a right order on G_v, for all $v \in V(Y)$. Then there is a common extension of all the $\leq_v$ to a right order on π if and only if there is a normal, compatible family $\{\mathcal{R}_v \mid v \in V(Y)\}$ for (G,Y) such that $\leq_v$ is in $\mathcal{R}_v$ for all $v \in V(Y)$, and $\varphi_e\varphi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(T)$.

Proof. Suppose π is right orderable and let $\mathcal{R}$ be the set of all right orders on π. For $v \in V(Y)$, let $\mathcal{R}_v$ be the set of right orders on G_v induced by orders in $\mathcal{R}$. Then $\mathcal{R}_v$ is normal as $\mathcal{R}$ is.

Let $e \in E(Y)$ and assume $\leq$ is in $\mathcal{R}_{o(e)}$. Let $\leq_\pi$ be one of its extensions belonging to $\mathcal{R}$. Then $\leq_\pi^{q_e}$ is in $\mathcal{R}$, and induces a right order on $G_{t(e)}$ by restriction; denote this order by $\leq_e$, so $\leq_e$ is in $\mathcal{R}_{t(e)}$.

Then, for $a \in G_e$, $1 \leq a^{\bar{e}}$ implies $1 \leq_\pi a^{\bar{e}} = q_e a^e q_e^{-1}$, that is, $1 \leq_\pi^{q_e} a^e$, hence $1 \leq_e a^e$. It follows that $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\leq, \leq_e)$, and so $\{\mathcal{R}_v \mid v \in V(Y)\}$ is compatible for (G,Y).

Suppose that $\leq_v$ is a right order on G_v, for all $v \in V(Y)$, and that there is a common extension of all the $\leq_v$ to a right order $\leq$ on π. Then π is right orderable, so $\mathcal{R}_v$, for $v \in V(Y)$, can be defined as above, and $\leq_v$ is in $\mathcal{R}_v$, for all $v \in V(Y)$. If $e \in E(T)$, then for $a \in G_e$,

$$1 \leq_{o(e)} a^{\bar{e}} \iff 1 \leq a^{\bar{e}} \iff 1 \leq a^e \iff 1 \leq_{t(e)} a^e$$

because in π, $a^{\bar{e}}$ and a^e are identified (the generator q_e of π is 1). Hence $\varphi_e \varphi_{\bar{e}}^{-1}$ $(a^{\bar{e}} \mapsto a^e)$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$.

Conversely, assume $\{\mathcal{R}_v \mid v \in V(Y)\}$ is a normal, compatible family of sets of right orders for (G,Y). Let F be the free group on $\{q_e \mid e \in A \setminus E(T)\}$. Let $\theta : \pi \longrightarrow F$ be the canonical epimorphism obtained by sending all elements of the vertex groups to 1, q_e to q_e, for $e \in A \setminus E(T)$, and q_e to 1 for $e \in E(T)$. Note that F can be viewed as $\pi(I,Y,T)$, where (I,Y) is the graph of groups with all edge and vertex groups trivial. By [2, Lemma 7], the quotient graph $Z = \widetilde{Y}/\ker(\theta)$ is a tree (isomorphic to $\widetilde{Y}(I,Y,T)$, which is the universal covering of the graph Y). An associated graph of groups (K,Z), with $\ker(\theta) \cong \pi(K,Z,Z)$, as in [8, §5.4, Ch.I], will be constructed as follows.

First note that there is an injective graph map $T \to \widetilde{Y}$, $x \mapsto \tilde{x}$, whose image is therefore a subtree $\widetilde{T}$ of $\widetilde{Y}$. Further, pairwise distinct vertices or edges of $\widetilde{T}$ are in different π-orbits, so in different $\ker(\theta)$-orbits. Therefore, if $p : \widetilde{Y} \longrightarrow Z$ is the projection map, its restriction to $\widetilde{T}$ is injective. By a simple modification of [8, §3.1, Proposition 14], there is a graph map $j : Z \to \widetilde{Y}$ with $pj = \mathrm{id}_Z$, and such that j maps $p(\widetilde{T})$ isomorphically onto $\widetilde{T}$. For a vertex or edge x of Z, put $K_x = \mathrm{stab}_{\ker(\theta)}(j(x))$, and let the monomorphism $\psi_e : K_e \longrightarrow K_{t(e)}$, for $e \in E(Z)$, be inclusion. This completes the construction of (K,Z), which is simpler than the general construction in [8] because Z is a tree (cf [8, §4.5, Ch.I]).

The stabilizers of vertices for the action of G on $\widetilde{Y}$ are conjugates of the vertex groups of (G,Y). Since $\ker(\theta)$ is normal in π, it contains all these stabilizers, and so all the edge stabilizers for the action of G. Thus $\mathrm{stab}_{\ker(\theta)}(x) = \mathrm{stab}_\pi(x)$ for any edge or vertex x of $\widetilde{Y}$, so this can be unambiguously written as $\mathrm{stab}(x)$.

For $u \in V(\widetilde{Y})$, define $\mathcal{R}_u$ to be the set of right orders in Lemma 3.1. Then, for $z \in V(Z)$, put $\mathcal{R}_z = \mathcal{R}_{j(z)}$, a normal set of right orders on K_z. For $e \in E(Z)$, $\psi_e \psi_{\bar{e}}^{-1}$ is the identity map on $K_e = \mathrm{stab}(j(e))$, so is compatible for $(\mathcal{R}_{o(e)}, \mathcal{R}_{t(e)})$ by Lemma 3.2, since $\mathcal{R}_{o(e)} = \mathcal{R}_{j(o(e))} = \mathcal{R}_{o(j(e))}$, and similarly $\mathcal{R}_{t(e)} = \mathcal{R}_{t(j(e))}$. By Lemma 2.3, $\ker(\theta)$ is right orderable.

Next, let $\leq_v$ belong to $\mathcal{R}_v$, for all $v \in V(Y)$, and assume that $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(T)$. A right order $\leq_u$ in $\mathcal{R}_u$ will be defined for every $u \in V(\widetilde{Y})$, such that, for all $f \in E(\widetilde{Y})$, id : stab$(f) \to$ stab(f) is compatible for $(\leq_{o(f)}, \leq_{t(f)})$. For $\tilde{v} \in V(\widetilde{T})$, let $\leq_{\tilde{v}}$ be $\leq_v$, an element of $\mathcal{R}_{\tilde{v}} = \mathcal{R}_v$. If $\tilde{e} \in E(\widetilde{T})$, so $e \in E(T)$, then $o(\tilde{e}) = \widetilde{o(e)}$, hence $\leq_{o(\tilde{e})} = \leq_{o(e)}$, and similarly $\leq_{t(\tilde{e})} = \leq_{t(e)}$. Since, for $a \in G_e$, a^e and $a^{\bar{e}}$ are identified in π, it follows that id : stab$(\tilde{e}) \to$ stab$(\tilde{e})$ is compatible for $(\leq_{o(\tilde{e})}, \leq_{t(\tilde{e})})$. (If $e \in A$ then stab$(\tilde{e}) = G_{\bar{e}}^{\bar{e}}$ and if $e \notin A$ then stab$(\tilde{e}) = G_e^e$.)

Now define $\leq_u$ for all $u \in V(\widetilde{Y})$ by induction on the distance in $\widetilde{Y}$ from $\widetilde{T}$ to u. If this distance is $n > 0$, there is a unique edge $f \in E(\widetilde{Y})$ such that $t(f) = u$ and $o(f)$ has distance $(n-1)$ from $\widetilde{T}$; assume $\leq_{o(f)}$ has been defined. Since, by Lemma 3.2, id : stab$(f) \to$ stab(f) is compatible for $(\mathcal{R}_{o(f)}, \mathcal{R}_{t(f)})$, $\leq_u$ can be chosen to be an element of $\mathcal{R}_{t(f)} = \mathcal{R}_u$ such that id : stab$(f) \to$ stab(f) is compatible for $(\leq_{o(f)}, \leq_u)$ (and so for $(\leq_u, \leq_{o(f)})$).

For $z \in V(Z)$, define $\leq_z$ to be $\leq_{j(z)}$, an element of $\mathcal{R}_z$. Then for $e \in E(Z)$, $\psi_e \psi_{\bar{e}}^{-1}$ is the identity map on stab$(j(e))$, so is compatible for $(\leq_{o(j(e))}, \leq_{t(j(e))})$. But $o(j(e)) = j(o(e))$, $t(j(e)) = j(t(e))$, so $\psi_e \psi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$. By Lemma 2.3, there is a right order $\leq$ on ker(θ) which induces $\leq_z$ on K_z, for all $z \in V(Z)$.

For $v \in V(Y)$, if $z = p(\tilde{v})$, then $\leq_z$ is by definition $\leq_{j(z)}$, $j(z) = \tilde{v}$ and by definition, $\leq_{\tilde{v}}$ is $\leq_v$. That is, $\leq_z$ is equal to $\leq_v$. Hence $\leq$ induces $\leq_v$ on $G_v = K_z$, for all $v \in V(Y)$.

There is an exact sequence

$$1 \longrightarrow \ker(\theta) \longrightarrow \pi \xrightarrow{\theta} F \longrightarrow 1.$$

Since F is free, it is right orderable, hence so is π, and any right order on ker(θ) extends to a right order on π (see eg [1, Lemma 2.1]). The theorem follows. □

There are several consequences of this theorem along the lines of §5 in [1]; just three will be given.

Corollary 3.4. *Let (G,Y) be a graph of groups, let T be a maximal tree of Y and let $\pi = \pi(G,Y,T)$. Suppose that G_v is right orderable, for all $v \in V(Y)$, and that every right order on G_e^e extends to a right order on $G_{t(e)}$, for all $e \in E(Y)$. Then π is right orderable.*

Let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups. If $\leq_v$ is a right order on G_v, for every $v \in V(Y)$, and $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(T)$, then there is a common extension of the $\leq_v$ to a right order on π.

Proof. For $v \in V(Y)$, let $\mathcal{R}_v$ be the set of all right orders on G_v, a normal set of right orders on G_v. Suppose $e \in E(Y)$, and $\leq$ is in $\mathcal{R}_{o(e)}$. Define a right order $\leq'$ on G_e^e by: $a \leq' b$ if and only if $\varphi_{\bar{e}} \varphi_e^{-1}(a) \leq \varphi_{\bar{e}} \varphi_e^{-1}(b)$. By assumption $\leq'$ extends to a right order on $G_{t(e)}$, which is in $\mathcal{R}_{t(e)}$. it follows that $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\mathcal{R}_{o(e)}, \mathcal{R}_{t(e)})$. The corollary now follows from Theorem 3.3. □

The next corollary is immediate from Corollary 3.4.

Corollary 3.5. *Let (G,Y) be a graph of groups, let T be a maximal tree of Y and let $\pi = \pi(G,Y,T)$. Suppose that $\leq_v$ is a right order on G_v, for all $v \in V(Y)$, and that G_e^e is a convex subgroup of $G_{t(e)}$, for all $e \in E(Y)$. Then π is right orderable.*

Let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups. If $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(T)$, then there is a common extension of the $\leq_v$ to a right order on π. □

Corollary 3.6. *Let (G,Y) be a graph of groups, let T be a maximal tree of Y and let $\pi = \pi(G,Y,T)$. Assume G_v is right orderable, for all $v \in V(Y)$, and G_e is cyclic, for all $e \in E(Y)$. Then π is right orderable.*

Let $\varphi_e : G_e \to G_{t(e)}$ be the monomorphism given by the graph of groups. If $\leq_v$ is a right order on G_v, for all $v \in V(Y)$, and $\varphi_e \varphi_{\bar{e}}^{-1}$ is compatible for $(\leq_{o(e)}, \leq_{t(e)})$, for all $e \in E(T)$, then there is a common extension of the $\leq_v$ to a right order on π.

Proof. There is exactly one right order on the trivial group, and exactly two on the infinite cyclic group, which are reverses of each other, so this follows from Corollary 3.4. □

Corollary 3.6 generalises Corollary 5.3 and Corollary 6.7 in [1]. Corollary 3.4 can also be used to prove the following, which is Theorem 2.12 in [4].

Theorem 3.7. *Let (G,Y) be a graph of groups, and let T be a maximal tree of Y. For $v \in V(Y)$, let $Z_v = \langle G_e^e \mid e \in E(Y) \text{ and } t(e) = v \rangle$. Assume*
(1) *G_e^e is central in $G_{t(e)}$, for all $e \in E(Y)$;*
(2) *for all $v \in V(Y)$, Z_v and Z_v/G_e^e, for all $e \in E(Y)$ such that $t(e) = v$, are torsion-free;*
(3) *G_v/Z_v is right orderable, for all $v \in V(Y)$.*

Then $\pi(G,Y,T)$ is right orderable.

Proof. Let $e \in E(Y)$; since $Z_{t(e)}/G_e^e$ is torsion-free abelian, it is (two-sided) orderable. There is an exact sequence

$$1 \longrightarrow G_e^e \longrightarrow Z_{t(e)} \longrightarrow Z_{t(e)}/G_e^e \longrightarrow 1$$

hence any right order on G_e^e extends to a right order on $Z_{t(e)}$ (see [1, Lemma 2.1]). For any $v \in V(Y)$, Z_v is torsion-free abelian, so orderable, and there is an exact sequence

$$1 \longrightarrow Z_v \longrightarrow G_v \longrightarrow G_v/Z_v \longrightarrow 1.$$

Since G_v/Z_v is right orderable, so is G_v, and any right order on Z_v extends to a right order on G_v, again by [1, Lemma 2.1]. By Corollary 3.4, $\pi(G,Y,T)$ is right orderable. □

The final applications are to group actions on Λ-trees, where Λ is a totally ordered abelian group. The theory of Λ-trees is discussed in [3], and some relevant points will be recalled here. A Λ-tree is a special kind of metric space (X,d), where the metric d takes values in Λ. The axioms are given in [3, §1, Chapter 2]. They imply that, for every x, $y \in X$, there is a unique isometry $\alpha : [0,c] \to X$ with $\alpha(0) = x$, $\alpha(c) = y$, where $c = d(x,y)$. Here, $[0,c] = \{a \in \Lambda \mid 0 \leq a \leq c\}$, and the metric d' on Λ is given by

$d'(a,b) = |a-b| := \max\{a-b, b-a\}$. The image of α is denoted by $[x,y]$ and is called a *segment*.

Isometries from X onto X are of three different kinds; *elliptic* (have a fixed point), *inversions* (g is an inversion if g has no fixed point but g^2 does), and *hyperbolic*. A hyperbolic isometry has an *axis*, which is metrically isomorphic to a convex subset of Λ, on which it acts as a translation. If g is a hyperbolic isometry, $\ell(g)$ is defined to be the amplitude of the translation on its axis, and if g is elliptic or an inversion, $\ell(g)$ is defined to be 0. Thus $\ell(g) \in \Lambda$, and is called the *hyperbolic length* of g. (See [3, Chapter 3].)

Actions of groups on Λ-trees will be by isometries. Thus a group G acts freely and without inversions on a Λ-tree if and only if every $g \in G \setminus \{1\}$ acts as a hyperbolic isometry, equivalently, $\ell(g) > 0$ for all $g \in G$, $g \neq 1$. If Λ is divisible, any action is necessarily without inversions (see criterion (iv) in Lemma 1.2, Chapter 3 in [3]).

The following is now an easy consequence of results in [6].

Theorem 3.8. *Suppose a group G acts freely on an $\mathbb{R}^n$-tree, where $\mathbb{R}^n$ has the lexicographic order. Then G is right orderable.*

Proof. Since a group is right orderable if and only if every finitely generated subgroup is right orderable, it suffices to show this under the extra hypothesis that G is finitely generated, and this will be done by induction on n. For $n = 1$, all groups acting freely on $\mathbb{R}$-trees are (two-sided) orderable. See Proposition 5.13, Chapter 5, and the remarks following it in [3]. Now assume it is true for $(n-1)$, and let G be a finitely generated group acting freely on an $\mathbb{R}^n$-tree. By Grushko's Theorem, G is a free product of (finitely many) finitely generated freely indecomposable groups, and a free product of right orderable groups is right orderable; this is well-known and a special case of Corollary 3.6 (cf [1, Corollary 5.11]). Thus it can be assumed that G is freely indecomposable. By [6, Theorem 7.1], G is the fundamental group of a finite graph of groups with cyclic edge groups, where each vertex group is finitely generated and has a free action on an $\mathbb{R}^{n-1}$-tree. By induction the vertex groups are right orderable, so by Corollary 3.6, G is right orderable. □

Remark. In Theorem 3.8, there is no need to specify the left or right lexicographic order on $\mathbb{R}^n$ as these give isomorphic ordered abelian groups.

Corollary 3.9. *Suppose a group G acts freely and without inversions on a Λ-tree, where Λ has only finitely many convex subgroups. Then G is right orderable.*

Proof. Since Λ has finitely many convex subgroups, it is isomorphic, as ordered abelian group, to $\Lambda_1 \oplus \ldots \oplus \Lambda_n$, with the lexicographic order, where each Λ_i is a subgroup of $\mathbb{R}$. See, for example, Theorem 1.2 and the proof of Lemma 1.6 in [3, Chapter 1]. Thus Λ can be assumed to be a subgroup of $\mathbb{R}^n$.

Now $\ell(g) > 0$ for all $g \in G$ with $g \neq 1$, where $\ell(g)$ is the hyperbolic length of g. By [3, Lemma 2.1, Chapter 3], G acts freely on the $\mathbb{R}^n$-tree $\mathbb{R}^n \otimes_\Lambda X$, so G is right orderable by Theorem 3.8. □

The corollary applies when Λ is a finitely generated ordered abelian group (see the remarks preceding Lemma 1.6 in [3]). A special case is when $\Lambda = \mathbb{Z}^n$ with the lexicographic order. For a discussion of groups acting freely and without inversions on a $\mathbb{Z}^n$-tree, see [7]; in particular, §3 contains examples of such groups when $n = 2$.

This can be extended a little further. Recall that, if $\{\Lambda_i \mid i \in I\}$ is a family of (totally) ordered abelian groups and I is linearly ordered, then $\bigoplus_{i\in I}\Lambda_i$ is an ordered abelian group via the right lexicographic order. Thus $a = (a_i)_{i\in I} > 0$ if $a_{i_0} > 0$, where i_0 is the greatest element of $\{i \in I \mid a_i \neq 0\}$. This applies to the additive group of the polynomial ring $\mathbb{R}[t]$ in one variable, which is $\bigoplus_{i\geq 0}\Lambda_i$, where $\Lambda_i = \mathbb{R}t^i \cong \mathbb{R}$, a direct sum of countably many copies of $\mathbb{R}$. Note that $\mathbb{R}^n$, with the right lexicographic order, may be viewed as a convex subgroup (the set of polynomials of degree at most $n-1$) and $\mathbb{R}[t] = \bigcup_{n\geq 1}\mathbb{R}^n$, an ascending union.

Corollary 3.10. *If G acts freely on an $\mathbb{R}[t]$-tree, then G is right orderable.*

Proof. It suffices to show this when G is finitely generated. Let (X,d) be an $\mathbb{R}[t]$-tree on which G acts freely, let $x_0 \in X$ and let $g_1,\dots,g_k$ be a finite generating set for G. If $g \in G$, then $g = g_{i_1}^{e_1}\dots g_{i_m}^{e_m}$ for some $m \geq 1$, where $e_j = \pm 1$ and $i_j \in \{1,\dots,k\}$ for $1 \leq j \leq m$. Then by the triangle inequality

$$\begin{aligned} d(x_0,gx_0) &\leq d(x_0,g_{i_1}^{e_1}x_0) + d(g_{i_1}^{e_1}x_0, g_{i_1}^{e_1}g_{i_2}^{e_2}x_0) + \dots + d(g_{i_1}^{e_1}\dots g_{i_{m-1}}^{e_{m-1}}x_0, gx_0) \\ &= d(x_0,g_{i_1}^{e_1}x_0) + d(x_0,g_{i_2}^{e_2}x_0) + \dots + d(x_0,g_{i_m}^{e_m}x_0) \\ &= d(x_0,g_{i_1}x_0) + d(x_0,g_{i_2}x_0) + \dots + d(x_0,g_{i_m}x_0). \end{aligned}$$

There exists n such that $d(x_0,g_1x_0),\dots,d(x_0,g_kx_0)$ all belong to $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a convex subgroup, it follows that $d(x_0,gx_0) \in \mathbb{R}^n$, hence $d(x_0,y) \in \mathbb{R}^n$ for all y in the segment $[x_0,gx_0]$, and all $g \in G$.

The subtree Y spanned by the G-orbit of x_0 is $\bigcup_{g\in G}[x_0,gx_0]$ (see the definition preceding Lemma 1.8, Chapter 2 in [3]), and is G-invariant. If y, $z \in Y$ then $d(y,z) \leq d(x_0,y) + d(x_0,z)$, hence $d(y,z) \in \mathbb{R}^n$. Thus Y is an $\mathbb{R}^n$-tree on which G acts freely, so G is right orderable by Theorem 3.8. □

Acknowledgment

The author thanks Vasily Bludov and Andrew Glass for their helpful comments.

REFERENCES

[1] V. V. Bludov and A. M. W. Glass, "Word problems, embeddings, and free products of right-ordered groups with amalgamated subgroup", *Proc. Lond. Math. Soc.* (3) **99** (2009), 585–608.
[2] I. M. Chiswell, "The Grushko-Neumann theorem", *Proc. Lond. Math. Soc.* (3) **33** (1976), 385–400.
[3] I. M. Chiswell, *Introduction to Λ-trees*. Singapore: World Scientific 2001.
[4] I. M. Chiswell, "Locally invariant orders on groups", *Internat. J. Algebra Comput.* **16** (2006), 1161–1179.
[5] D. E. Cohen, *Combinatorial group theory: a topological approach*, London Math. Soc. Stud. Texts **14**. Cambridge: University Press 1989.
[6] V. Guirardel, "Limit groups and groups acting freely on $\mathbb{R}^n$-trees", *Geom. Topol.* **8** (2004), 1427–1470.

[7] A. Martino and S. O'Rourke, "Some free actions on non-Archimedean trees", *J. Group Theory* **7** (2004), 275–286.
[8] J.-P. Serre, *Trees.* New York: Springer 1980.

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