ORDERING FREE PRODUCTS OF GROUPS

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To Professor Ján Jakubík on the occasion of his 90th birthday.

Abstract. A method of constructing orders on free products of groups is given, based on work of Botto Mura and Rhemtulla, and of Holland and Medvedev.

1. Introduction

In the book of Botto Mura and Rhemtulla [5, Theorem 2.3.1], there is an argument, attributed to P. Hall, to prove the theorem of Vinogradov [6], that a free product of ordered groups can be ordered. It is pointed out in Holland and Medvedev [3] that the argument as given there does not work. However, the argument does show that, given an order on a free product, more generally a sequence of orders, a new order can be constructed. The details were carried out, in the context of free groups, in [3], with some interesting consequences. In fact, Holland and Medvedev went further, carrying out their construction using only partial orders, and by iterating, they were able to construct orders on free groups without an initial sequence of total orders.

One of several methods of ordering free groups is that given by Bergman [1]. This was adapted for free products of ordered groups in [2]. Given a family of ordered groups and a total order on the index set, this gives a canonical way of ordering the free product of the family. This was expressed in [2] by defining an appropriate category and a functor from this category to the category of ordered groups and order-preserving homomorphisms, which will be denoted by $\mathbf{O}$.

The method of Bergman is formally similar, although different from the argument in [5]. Here a version of the construction in [3] will be carried out in the context of free products, and in the style of [2]. This entails defining a category whose objects are unfortunately rather elaborate, and in general the basic construction does not appear to define even a partial order on a free product of groups. However, it does work to define a total order when all the initial orders involved are total orders. This leads to sufficient conditions for the procedure to

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define a partial order, and this result is enough for the construction of orders on free products without an initial sequence of total orders. The basic construction is carried out in the next section, and a normal form, analogous to that in [3], is considered in the following section. Then a version of the iterative procedure of [3] is carried out in Section 4, to construct total orders on a free product of ordered groups, given a total order on the index set.

To fix terminology, some basic definitions will be recalled. A partial right order on a group G is a partial order $\leq$ on G such that, for all x, y and $z \in G$, $x \leq y$ implies $xz \leq yz$. Similarly, a partial left order on a group G is a partial order $\leq$ on G such that, for all x, y and $z \in G$, $x \leq y$ implies $zx \leq zy$, and a partial order that satisfies both of these conditions is called a two-sided partial order, or just a partial order on G. If the order is total, the word 'partial' is omitted (or sometimes replaced by 'total' for emphasis). Thus an order on a group G is a total order satisfying both conditions. An ordered group is a group G together with an order on G.

Given a partial right order $\leq$ on a group G, the *strictly positive cone* is the set $P := \{x \in G \mid 1 < x\}$. It has the properties

$$P\,P \subset P, \quad P \cap P^{-1} = \emptyset.$$

Conversely, given a subset of G satisfying these conditions, a partial right order can be defined on G by: $x \leq y$ if and only if $x = y$ or $yx^{-1} \in P$, and P is the strictly positive cone for this partial right order. Further, it is a (two-sided) partial order if and only if $x^{-1}Px \subseteq P$ for all $x \in G$, and it is a total order if and only if $G \setminus \{1_G\} = P \cup P^{-1}$. In particular, the strictly positive cone of a partial order on a group G is a normal subsemigroup of G. Note that the trivial partial order ($x \leq y$ if and only if $x = y$) gives a partial order on any group G, whose strictly positive cone is empty. In what follows, the term "strictly positive cone" will be abbreviated to "positive cone".

2. New Orders for Old

As indicated in the introduction, a category F will be defined, and a construction given which applies to objects of F, which in certain circumstances will lead to an order or partial order on a free product of groups.

The objects of F are triples $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ where G is a group, $\{G_\lambda \mid \lambda \in \Lambda\}$ is a family of ordered subgroups of G, $\leq_0$ is a partial order on the set Λ, $\leq_i$ is a partial order on the group G, for all $i \in \mathbb{N}_{>0}$, and $G = \ast_{\lambda \in \Lambda} G_\lambda$ (that is, G is the free product of its family of subgroups $\{G_\lambda \mid \lambda \in \Lambda\}$). Note that the orders $\leq_i$ can be arbitrary, in particular, the restriction of $\leq_i$ to G_λ, for $i > 0$, $\lambda \in \Lambda$, need not be related in any way to the given order on G_λ. It should also be emphasised that the given order on G_λ is a total order.

The set Λ is called the *index set* of $\mathcal{G}$ and G_λ is called a *free factor* of $\mathcal{G}$. Also, G is denoted by $\langle\mathcal{G}\rangle$. The strict order corresponding to $\leq_i$ will be denoted by $<_i$.

Let

$$\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\}) \text{ and } \mathcal{H} = (H, \{H_\mu \mid \mu \in M\}, \{\preceq_i \mid i \in \mathbb{N}\})$$

be objects of F. An F-morphism from $\mathcal{G}$ to $\mathcal{H}$ is a pair

$$\mathsf{f} = (\varphi, \{f_\lambda \mid \lambda \in \Lambda\})$$

where $\varphi : \Lambda \to M$ is an order isomorphism, and for each $\lambda \in \Lambda$, $f_\lambda : G_\lambda \to H_{\lambda\varphi}$ is an isomorphism of ordered groups, such that for all $i \in \mathbb{N}_{>0}$ and $g_1, g_2 \in G$, $g_1 \leq_i g_2$ if and only if $g_1\bar{\mathsf{f}} \preceq_i g_2\bar{\mathsf{f}}$, where $\bar{\mathsf{f}}$ is the unique extension of the f_λ to an isomorphism $G \to H$.

If $\mathsf{g} = (\psi, \{g_\mu \mid \mu \in M\}) : \mathcal{H} \to \mathcal{K}$ is a morphism, fg is defined to be the morphism $(\varphi\psi, \{f_\lambda g_{\lambda\varphi} \mid \lambda \in \Lambda\})$, and the identity morphism $1_{\mathcal{G}}$ is defined to be $(id_\Lambda, \{id_{G_\lambda} \mid \lambda \in \Lambda\})$. Clearly this makes F into a category.

The first aim is to define a subset $P_{\mathcal{G}}$ of $\langle\mathcal{G}\rangle$, which in certain circumstances will be the positive cone for a partial or total order on $\langle\mathcal{G}\rangle$. To do this, an auxiliary construction is necessary. Let $\nu \in \Lambda$, and define $\Lambda_\nu := \ast_{\nu <_0 \lambda} G_\lambda$. Then set

$$L = \langle a^{-1}G_\nu a \mid a \in \Lambda_\nu\rangle.$$

Then $L = \ast_{a\in\Lambda_\nu} a^{-1}G_\nu a$. To see this, if $u = a_1^{-1}g_1a_1 \dots a_n^{-1}g_na_n$, where $a_j \in \Lambda_\nu$, $g_j \in G_\nu \setminus \{1\}$, for $1 \leq j \leq n$ and $a_j \neq a_{j+1}$ for $1 \leq j < n$, then viewing this as a word in $\bigcup_{\lambda\in\Lambda} G_\lambda$ and cancelling / consolidating to obtain a reduced word, the letters g_j $(1 \leq j \leq n)$, the initial word a_1^{-1} and the final word a_n remain. This follows by induction on n. Thus $u \neq 1$, hence L is a free product as claimed.

Define an order on $a^{-1}G_\nu a$ by: $a^{-1}ga < a^{-1}ha$ if and only if $g < h$ in G_ν. Then let $\leq_0'$ be the restriction of $\leq_1$ to Λ_ν. This gives a new object in F, namely

$$\mathcal{G}_\nu := (L, \{G_{\nu,a} \mid a \in \Lambda_\nu\}, \{\leq_i' \mid i \in \mathbb{N}\})$$

where, for $a \in \Lambda_\nu$, $G_{\nu,a} = a^{-1}G_\nu a$, and for $i \in \mathbb{N}_{>0}$, $\leq_i'$ is the restriction to L of $\leq_{i+1}$. Thus $L = \langle\mathcal{G}_\nu\rangle$.

Remark 2.1. Strictly, the index set is $\{\nu\} \times \Lambda_\nu$, ordered via the ordering on Λ_ν, but in what follows it will cause no confusion to view Λ_ν as the index set.

Take $1 \neq g \in G$ and write g as a reduced word relative to the decomposition $\ast_{\lambda\in\Lambda} G_\lambda$, say $g = g_1 \dots g_k$, where $g_j \in G_{\lambda_j}$. The $\mathcal{G}$-*length* of g is defined to be k.

If the set $\{\lambda_j \mid 1 \leq j \leq k\}$ has a least element, it will be denoted by $gm_{\mathcal{G}}$. Before proceeding, here are some properties of the function $m_{\mathcal{G}}$ which will be used later.

Remark 2.2. (1) If $\nu = hm_{\mathcal{G}}$ and $g \in \ast_{\nu <_0 \lambda} G_\lambda = \Lambda_\nu$, then $(gh)m_{\mathcal{G}} = \nu$.
(2) If $1 \neq h \in \langle \mathcal{G}_\nu \rangle$, then $hm_{\mathcal{G}} = \nu$.

The proofs are easy and left to the reader ((2) follows from the discussion above showing that L is a free product). Suppose $gm_{\mathcal{G}}$ is defined, and denote it by ν. Rewrite $g = g_1 \dots g_k$ as $g = a_0 b_1 a_1 \dots b_n a_n$, where $a_j \in G_\nu$, and $b_j \in \ast_{\nu <_0 \lambda} G_\lambda = \Lambda_\nu$, which in turn can be rewritten as

$$g = (b_1 \dots b_n) \prod_{j=0}^{n} (b_{j+1} \dots b_n)^{-1} a_j (b_{j+1} \dots b_n) = g' g^* \tag{2.1}$$

where $g' = b_1 \dots b_n \in \Lambda_\nu$ and $g^* \in \langle \mathcal{G}_\nu \rangle$. This decomposition is unique: if $g = h' h^*$, where $h' \in \Lambda_\nu$ and $h^* \in \langle \mathcal{G}_\nu \rangle$, then $g\,p = g' = h'$, where $p : G \to \Lambda_\nu$ is the projection map, (which is trivial on G_λ for $\lambda \leq \nu$, and the identity on G_λ for $\lambda > \nu$). Hence also $g^* = h^*$.

Remark 2.3. If $g \neq 1$ then $g^* \neq 1$, and if $g \notin G_\nu$ then the $\mathcal{G}_\nu$-length of g^* is less than the $\mathcal{G}$-length of g.

Now define a subset $X_{\mathcal{G}}$ of $\langle \mathcal{G} \rangle$ recursively as follows.

(1) If $gm_{\mathcal{G}}$ is not defined, then $g \notin X_{\mathcal{G}}$.
(2) If $g \in G_\nu$ then $g \in X_{\mathcal{G}}$ if and only if $g > 1$ in the given order on G_ν.
(3) If $g' \neq 1$ (so $g \notin G_\nu$), then g' has shorter $\mathcal{G}$-length than g, and $g \in X_{\mathcal{G}}$ if and only if $g' \in X_{\mathcal{G}}$.
(4) If $g \notin G_\nu$ and $g' = 1$, then g^* has $\mathcal{G}_\nu$-length shorter that the $\mathcal{G}$-length of g, and $g \in X_{\mathcal{G}}$ if and only if $g^* \in X_{\mathcal{G}_\nu}$.

Define $P_{\mathcal{G}}$ to be the *normal* subsemigroup of G generated by $X_{\mathcal{G}}$.

Note that, if $gm_{\mathcal{G}}$ is defined, then

$$\left.\begin{aligned} g^{-1} m_{\mathcal{G}} = gm_{\mathcal{G}},\ g^{-1} = (g')^{-1}(g'(g^*)^{-1}(g')^{-1}) \\ \text{and } (g')^{-1} \in \Lambda_\nu,\ g'(g^*)^{-1}(g')^{-1} \in \langle \mathcal{G}_\nu \rangle,\ \text{where } \nu = gm_{\mathcal{G}}. \end{aligned}\right\} \tag{2.2}$$

It follows easily by induction on length that for all $g \in G$, at most one of g, $g^{-1} \in X_{\mathcal{G}}$, that is, $X_{\mathcal{G}} \cap X_{\mathcal{G}}^{-1} = \emptyset$. However, there is no obvious reason for $P_{\mathcal{G}} \cap P_{\mathcal{G}}^{-1} = \emptyset$ to be true. Before considering situations where this is true, it will be shown that the sets $X_{\mathcal{G}}$ are preserved by morphisms.

Lemma 2.1. *Let*

$$\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\}),\ \mathcal{H} = (H, \{H_\mu \mid \mu \in M\}, \{\preceq_i \mid i \in \mathbb{N}\})$$

be objects of $\mathbf{F}$, *and let* $\mathbf{f} = (\varphi, \{f_\lambda \mid \lambda \in \Lambda\})$ *be a morphism from* $\mathcal{G}$ *to* $\mathcal{H}$. *Then* $X_{\mathcal{G}}\bar{\mathbf{f}} \subseteq X_{\mathcal{H}}$; *consequently,* $P_{\mathcal{G}}\bar{\mathbf{f}} \subseteq P_{\mathcal{H}}$.

Proof. Firstly, for $\nu \in \Lambda$, f induces a morphism $\mathsf{f}_\nu = (\psi_\nu, \{f_a \mid a \in \Lambda_\nu\})$ from $\mathcal{G}_\nu$ to $\mathcal{H}_{\nu\varphi}$, as follows. For $a \in \Lambda_\nu$, define $a\psi_\nu = a\bar{\mathsf{f}}$. Note that $a\bar{\mathsf{f}} \in \ast_{\nu\varphi \prec_0 \mu} H_\mu = M_{\nu\varphi}$, because φ is order-preserving. Also, $\psi_\nu : \Lambda_\nu \to M_{\nu\varphi}$ maps Λ_ν bijectively onto $M_{\nu\varphi}$, and by the conditions for a morphism, it is order-preserving.

For $a \in \Lambda_\nu$, the mapping $f_a : a^{-1}G_\nu a \to (a\bar{\mathsf{f}})^{-1}H_{\nu\varphi}(a\bar{\mathsf{f}})$ is defined by $a^{-1}ga \mapsto (a\bar{\mathsf{f}})^{-1}(gf_\nu)(a\bar{\mathsf{f}})$. This is clearly an isomorphism of ordered groups, as f_ν is order-preserving.

Note that f_a is $\bar{\mathsf{f}}$ restricted to $a^{-1}G_\nu a$, and it follows that $\bar{\mathsf{f}}_\nu$ is $\bar{\mathsf{f}}$ restricted to $\langle \mathcal{G}_\nu \rangle$. Thus, for $g_1, g_2 \in \langle \mathcal{G}_\nu \rangle$ and $i \in \mathbb{N}_{>0}$,

$$g_1 \le_i' g_2 \Rightarrow g_1 \le_{i+1} g_2 \Rightarrow g_1\bar{\mathsf{f}} \preceq_{i+1} g_2\bar{\mathsf{f}} \Rightarrow g_1\bar{\mathsf{f}}_\nu \preceq_i' g_2\bar{\mathsf{f}}_\nu,$$

hence f_ν is indeed a morphism of F.

To prove the lemma, it will be shown, by induction on n, that for all n and any morphism $\mathsf{f} = (\varphi, \{f_\lambda \mid \lambda \in \Lambda\})$ of F, say from $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\le_i \mid i \in \mathbb{N}\})$ to $\mathcal{H} = (H, \{H_\mu \mid \mu \in M\}, \{\preceq_i \mid i \in \mathbb{N}\})$, if $g \in \langle \mathcal{G} \rangle$, $g \neq 1$, has $\mathcal{G}$-length n, then $g \in X_\mathcal{G}$ implies $g\bar{\mathsf{f}} \in X_\mathcal{H}$.

Assume then, that g has $\mathcal{G}$-length n and $g \in X_\mathcal{G}$. Write $g = g'g^*$ as in the recursive definition, so $g' \in \ast_{\nu <_0 \lambda} G_\lambda$, $g^* \in \langle \mathcal{G}_\nu \rangle$, where $\nu = gm_\mathcal{G}$. Then

$$\begin{aligned} h := g\bar{\mathsf{f}} &= (g'\bar{\mathsf{f}})\,(g^*\bar{\mathsf{f}}) \\ &= (g'\bar{\mathsf{f}})\,(g^*\bar{\mathsf{f}}_\nu) \end{aligned}$$

and $g'\bar{\mathsf{f}} \in \ast_{\nu\varphi \prec_0 \mu} H_\mu$, $g^*\bar{\mathsf{f}}_\nu \in \langle \mathcal{H}_{\nu\varphi} \rangle$. Let $g = g_1 \dots g_n$ be the expression of g as a reduced word relative to the decomposition $\ast_{\lambda \in \Lambda} G_\lambda$, where $g_k \in G_{\lambda_k}$. Then $h = h_1 \dots h_n$ is the expression of h as a reduced word relative to the decomposition $\ast_{\mu \in M} H_\mu$, where $h_k = g_k f_{\lambda_k} \in H_{\lambda_k\varphi}$. Since φ is order-preserving, it follows that $hm_\mathcal{H} = \nu\varphi$. Therefore, $h' = g'\bar{\mathsf{f}}$ and $h^* = g^*\bar{\mathsf{f}}_\nu$. There are three possibilities.

(1) If $g \in G_\nu$ then $g > 1$ in G_ν, and $h = gf_\nu$, hence $h > 1$ in $H_{\nu\varphi}$ since f_ν is order-preserving, and by definition $h \in X_\mathcal{H}$.
(2) If $g' \neq 1$, then g' has shorter $\mathcal{G}$-length than g, and $g' \in X_\mathcal{G}$, so $h' \in X_\mathcal{H}$ by induction, hence $h \in X_\mathcal{H}$ by definition.
(3) If $g' = 1$ and $g \notin G_\nu$, then $h' = 1$ and g^* has shorter $\mathcal{G}_\nu$-length than the $\mathcal{G}$-length of g so by induction and the definition of $X_\mathcal{G}$, $X_\mathcal{H}$,

$$g \in X_\mathcal{G} \Rightarrow g^* \in X_{\mathcal{G}_\nu} \Rightarrow h^* = g^*\bar{\mathsf{f}}_\nu \in X_{\mathcal{H}_{\nu\varphi}} \Rightarrow h \in X_\mathcal{H}.$$

This completes the proof that $X_\mathcal{G}\bar{\mathsf{f}} \subseteq X_\mathcal{H}$, and it follows that $P_\mathcal{G}\bar{\mathsf{f}} \subseteq P_\mathcal{H}$ since $\bar{\mathsf{f}}$ is a group homomorphism. □

Definition. Denote by F_0 the full subcategory of F whose objects are the objects of F of the form

$$\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\le_i \mid i \in \mathbb{N}\})$$

where $\le_i$ is a total order, for all $i \in \mathbb{N}$.

Next it will be shown that, for objects $\mathcal{G}$ of F_0, the construction works well and $P_{\mathcal{G}} = X_{\mathcal{G}}$ is the positive cone for an order on $\langle \mathcal{G} \rangle$. This is the situation considered in [5], indeed the proof of the next two lemmas is essentially the argument for [5, Theorem 2.3.1], although as presented there the argument does not work.

If $\mathcal{G}$ is an object of F_0, then it is easily seen by induction on $\mathcal{G}$-length, using equations 2.2, that for all $g \in G \setminus \{1_G\}$ (where $G = \langle \mathcal{G} \rangle$), either $g \in X_{\mathcal{G}}$ or $g^{-1} \in X_{\mathcal{G}}$; that is, $G \setminus \{1_G\} = X_{\mathcal{G}} \cup X_{\mathcal{G}}^{-1}$.

Lemma 2.2. *Let $\mathcal{G}$ be an object of F_0. If g, $h \in X_{\mathcal{G}}$ then $gh \in X_{\mathcal{G}}$.*

Proof. Let $\nu = gm_{\mathcal{G}}$ and $\kappa = hm_{\mathcal{G}}$. Use induction on the sum of the $\mathcal{G}$-lengths of g and h.

Case 1. $\nu < \kappa$. Then $gh = g'h(h^{-1}g^*h)$, so $(gh)' = g'h$, and $g'h \in X_{\mathcal{G}}$, either because $g' = 1$ or because $g' \in X_{\mathcal{G}}$ and g' has smaller $\mathcal{G}$-length than g and the induction hypothesis applies.

Case 2. $\kappa < \nu$. Similarly $gh = gh'$ and either $h' = 1$ or the induction hypothesis applies.

Case 3. $\kappa = \nu$. Then

$$gh = g'g^*h'h^* = (g'h')((h'^{-1}g^*h')h^*)$$

whence $(gh)' = g'h'$. If $g' \neq 1$ or $h' \neq 1$ then $(gh)' \in X_{\mathcal{G}}$, so $gh \in X_{\mathcal{G}}$. Otherwise $(gh)' = 1$ and $gh = (gh)^* = g^*h^*$, and the sum of the $\mathcal{G}_\nu$-lengths of g^* and h^* is less than the sum of the $\mathcal{G}$-lengths of g and h. Hence by induction, $(gh)^* \in X_{\mathcal{G}_\nu}$, so by definition $gh \in X_{\mathcal{G}}$.

This completes the proof. □

Thus if $\mathcal{G}$ is an object of F_0, $X_{\mathcal{G}}$ is the positive cone for a right order on $\langle \mathcal{G} \rangle$.

Lemma 2.3. *Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of F_0 and let $x \in \langle \mathcal{G} \rangle$. Then $x^{-1}X_{\mathcal{G}}x \subseteq X_{\mathcal{G}}$.*

Proof. It will be shown that, for all $\mathcal{G}$, all $x \in \langle \mathcal{G} \rangle$ and all $g \in X_{\mathcal{G}}$ of $\mathcal{G}$-length n, $x^{-1}gx \in X_{\mathcal{G}}$, by induction on n. Since $\langle \mathcal{G} \rangle$ is generated by $\bigcup_{\lambda \in \Lambda} G_\lambda$, it can be assumed that $x \in G_\kappa$ for some κ. Write $g = g'g^*$ as in the recursive definition, so $g' \in \ast_{\nu <_0 \lambda} G_\lambda$, $g^* \in \langle \mathcal{G}_\nu \rangle$, where $\nu = gm_{\mathcal{G}}$.

Case 1. $\kappa < \nu$. Then $(x^{-1}gx)' = g$, so if $g \in X_{\mathcal{G}}$ then by definition $x^{-1}gx \in X_{\mathcal{G}}$.

Case 2. $\kappa = \nu$. Then $(x^{-1}gx)' = g'$, so if $g' \neq 1$ then

$$g \in X_{\mathcal{G}} \Rightarrow g' \in X_{\mathcal{G}} \Rightarrow (x^{-1}gx)' \in X_{\mathcal{G}} \Rightarrow x^{-1}gx \in X_{\mathcal{G}}.$$

If $g' = 1$ then $x^{-1}gx = (x^{-1}gx)^* = x^{-1}g^*x$, $x \in \langle \mathcal{G}_\nu \rangle$ and $g = g^*$ has shorter $\mathcal{G}_\nu$-length than the $\mathcal{G}$-length of g, so by induction

$$g \in X_{\mathcal{G}} \Rightarrow g \in X_{\mathcal{G}_\nu} \Rightarrow x^{-1}gx \in X_{\mathcal{G}_\nu} \Rightarrow x^{-1}gx \in X_{\mathcal{G}}.$$

Case 3. $\kappa > \nu$. Then $(x^{-1}gx)' = x^{-1}g'x$, so if $g' \neq 1$, then g' has shorter $\mathcal{G}$-length than g and the induction hypothesis applies.

Suppose $g' = 1$. Then conjugation by x induces a morphism

$$\mathsf{f} = (\varphi, \{f_a \mid a \in \Lambda_\nu\}) : \mathcal{G}_\nu \to \mathcal{G}_\nu,$$

where $a\varphi = ax$ and $yf_a = x^{-1}yx$ for $y \in a^{-1}G_\nu a$. Note that φ is order-preserving, since $\leq_1$ is an order on G, so in particular a right order. Also, for $i > 0$, $\leq_i'$ is preserved by conjugation by x, since it is obtained by restriction from $\leq_{i+1}$, which is an order on G. Hence f is indeed a morphism. [1] By Lemma 2.1,

$$g \in X_\mathcal{G} \Rightarrow g = g^* \in X_{\mathcal{G}_\nu} \Rightarrow x^{-1}gx = g\bar{\mathsf{f}} \in X_{\mathcal{G}_\nu} \Rightarrow x^{-1}gx \in X_\mathcal{G}$$

since $x^{-1}gx = (x^{-1}gx)^*$.

This completes the inductive proof. □

The last two lemmas establish the following.

Proposition 2.4. *If $\mathcal{G}$ is an object of F_0, then $X_\mathcal{G} = P_\mathcal{G}$ is the positive cone for an order on $\langle\mathcal{G}\rangle$.* □

For an object $\mathcal{G}$ of F_0, define $\mathcal{G}Q$ to be $\langle\mathcal{G}\rangle$ with the order defined by $P_\mathcal{G}$, and for a morphism f, define $\mathsf{f}Q$ to be $\bar{\mathsf{f}}$. It is easily checked that this defines a functor $Q : \mathsf{F}_0 \to \mathbf{O}$.

In general, there seems no reason why $P_\mathcal{G}$ should be the positive cone for even a partial order on $\langle\mathcal{G}\rangle$. However, this proposition can be used to show that, for certain objects of F, $P_\mathcal{G}$ is the positive cone for a partial order. Before stating this result, it is convenient to define a partial order on the class of objects of F and establish some of its properties.

Definition. Let

$$\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\}),\ \mathcal{H} = (H, \{H_\mu \mid \mu \in M\}, \{\preceq_i \mid i \in \mathbb{N}\})$$

be objects of F. Then $\mathcal{G} \leq \mathcal{H}$ means that $G \leq H$, $\Lambda \subseteq M$, $\preceq_i$ is an extension of $\leq_i$, for all $i \in \mathbb{N}$, and for $\lambda \in \Lambda$, $H_\lambda = G_\lambda$ (as ordered group).

Clearly this defines a partial order on the class of objects of F.

Lemma 2.5. *Let*

$$\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\}),\ \mathcal{H} = (H, \{H_\mu \mid \mu \in M\}, \{\preceq_i \mid i \in \mathbb{N}\})$$

be objects of F, and suppose $\mathcal{G} \leq \mathcal{H}$. Then

(1) *For $\nu \in \Lambda$, $\mathcal{G}_\nu \leq \mathcal{H}_\nu$;*
(2) *$X_\mathcal{G} \subseteq X_\mathcal{H}$, hence $P_\mathcal{G} \subseteq P_\mathcal{H}$.*

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[1] It is essential that $\leq_{i+1}$ is an order, not just a right order, and this is why the argument in [5] fails.

Proof. (1) For $\nu \in \Lambda$,

$$\Lambda_\nu = \mathop{\ast}_{\nu <_0 \lambda} G_\lambda \leq \mathop{\ast}_{\nu \prec_0 \mu} H_\mu = M_\nu$$

where $\lambda \in \Lambda$, $\mu \in M$, and (1) follows easily.

(2) It will be shown by induction on the $\mathcal{G}$-length of g that if $g \in X_\mathcal{G}$, then $g \in X_\mathcal{H}$. If $g \in G_\nu$ for some $\nu \in \Lambda$, then $g > 1$ in $G_\nu = H_\nu$, hence $g \in X_\mathcal{H}$.

Otherwise, g has a decomposition as $g = g'g^*$, where $g' \in \ast_{\nu <_0 \lambda} G_\lambda$, $g^* \in \langle \mathcal{G}_\nu \rangle$, and $\nu = gm_\mathcal{G} = gm_\mathcal{H}$. Then $g' \in \ast_{\nu \prec_0 \mu} H_\mu$, and by (1), $g^* \in \langle \mathcal{H}_\nu \rangle$.

If $g' \neq 1$ then $g' \in X_\mathcal{G}$, and has shorter $\mathcal{G}$-length than g, so by induction $g' \in X_\mathcal{H}$, hence $g \in X_\mathcal{H}$ by definition of $X_\mathcal{H}$.

If $g' = 1$, then $g^* \in X_{\mathcal{G}_\nu}$, and the $\mathcal{G}_\nu$-length of g^* is smaller than the $\mathcal{G}$-length of g, so by induction and (1), $g^* \in X_{\mathcal{H}_\nu}$, hence $g \in X_\mathcal{H}$ by definition of $X_\mathcal{H}$. Thus $X_\mathcal{G} \subseteq X_\mathcal{H}$, and it follows that $P_\mathcal{G} \subseteq P_\mathcal{H}$. □

Corollary 2.6. *Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of F. Suppose $\leq_i$ can be extended to a total order $\preceq_i$ on G, for all $i > 0$. Then $P_\mathcal{G}$ is the positive cone for a partial order on G, which can be extended to an order on G.*

Proof. Define

$$\overline{\mathcal{G}} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\preceq_i \mid i \in \mathbb{N}\})$$

where $\preceq_0$ is an extension of $\leq_0$ to a total order on the set Λ, so $\overline{\mathcal{G}}$ is an object of F_0. Then $\mathcal{G} \leq \overline{\mathcal{G}}$, so $P_\mathcal{G} \subseteq P_{\overline{\mathcal{G}}}$, and $P_{\overline{\mathcal{G}}}$ is the positive cone for an order on G by Proposition 2.4. Hence $P_\mathcal{G} \cap P_\mathcal{G}^{-1} \subseteq P_{\overline{\mathcal{G}}} \cap P_{\overline{\mathcal{G}}}^{-1} = \emptyset$, so $P_\mathcal{G}$ is the positive cone for a partial order on G, and this partial order is extended by the order defined by $P_{\overline{\mathcal{G}}}$. □

3. Normal Form

Given an object $\mathcal{G}$ of F, a normal form will be established for the elements of $\langle \mathcal{G} \rangle$. Although the normal form will be shown to be unique, if it exists, there is no guarantee, in general, that an element will have a normal form.

Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of F and let $\lambda_0 \in \Lambda$. One can form $\mathcal{G}_{\lambda_0}$, with index set Λ_{λ_0}. Given $\lambda_1 \in \Lambda_{\lambda_0}$, the construction can be repeated, obtaining $(\mathcal{G}_{\lambda_0})_{\lambda_1}$ with index set $(\Lambda_{\lambda_0})_{\lambda_1}$. Continuing (and omitting parentheses) gives an object $\mathcal{G}_{\lambda_0 \dots \lambda_n}$ of F with index set $\Lambda_{\lambda_0 \dots \lambda_n}$.

Definition. A sequence of indices arising in this way, padded with 1's to give an infinite sequence $(\lambda_0, \dots, \lambda_n, 1, 1, \dots)$ is called a *$\mathcal{G}$-descent sequence.*

For later use, note that, by an easy induction, $\langle \mathcal{G}_{\lambda_0 \dots \lambda_n} \rangle = \mathop{\ast}_{\lambda \in \Lambda_{\lambda_0 \dots \lambda_n}} G_{\lambda_0}^{\lambda_1 \dots \lambda_n \lambda}$ and

$$\Lambda_{\lambda_0 \dots \lambda_n} = \mathop{\ast}_{\lambda_n <_n \lambda} G_{\lambda_0}^{\lambda_1 \dots \lambda_{n-1} \lambda}, \text{ where } \lambda \in \Lambda_{\lambda_0 \dots \lambda_{n-1}}. \tag{3.1}$$

(When $n = 0$, $\Lambda_{\lambda_0\ldots\lambda_{n-1}}$ is to be interpreted as Λ, and $G_{\lambda_0}^{\lambda_1\ldots\lambda_{n-1}\lambda}$ as G_λ.) Strictly (cf Remark 2.1), the index set of $\mathcal{G}_{\lambda_0\ldots\lambda_n}$ is $\{(\lambda_0,\ldots,\lambda_n)\}\times\Lambda_{\lambda_0\ldots\lambda_n}$, and for $\lambda\in\Lambda_{\lambda_0\ldots\lambda_n}$, the free factor $G_{\lambda_0\ldots\lambda_n\lambda}$ of $\mathcal{G}_{\lambda_0\ldots\lambda_n}$ is $G_{\lambda_0}^{\lambda_1\ldots\lambda_n\lambda}$. However, to keep notation as simple as possible, the index set will be viewed as $\Lambda_{\lambda_0\ldots\lambda_n}$.

Let $G_i = G$ with the ordering $\leq_i$. Then a $\mathcal{G}$-descent sequence is an element of the set $\Lambda\times G_1\times G_2\times\ldots$, and this set can be partially ordered lexicographically, hence the set of $\mathcal{G}$-descent sequences is partially ordered by restriction. For a $\mathcal{G}$-descent sequence $\boldsymbol{\lambda} = (\lambda_0,\ldots,\lambda_n,1,1,\ldots)$, define, for $g\in G_{\lambda_0}$,

$$g^{\boldsymbol{\lambda}} = g^{\lambda_1\ldots\lambda_n}.$$

(As usual, $g^{\lambda_1\ldots\lambda_n}$ means $(\lambda_1\ldots\lambda_n)^{-1}g(\lambda_1\ldots\lambda_n)$. If $g\notin G_{\lambda_0}$, $g^{\boldsymbol{\lambda}}$ is undefined.)

Remark 3.1. If $\boldsymbol{\lambda} = (\lambda_0,\lambda_1,\ldots,\lambda_n,1,1,\ldots)$ is a $\mathcal{G}$-descent sequence and $1\neq h\in G_{\lambda_0}$, then $h^{\boldsymbol{\lambda}}m_{\mathcal{G}} = \lambda_0$.

Proof. Because $\lambda_1 \in \ast_{\lambda_0<_0\lambda} G_\lambda$, $h^{\lambda_1}\in\langle\mathcal{G}_{\lambda_0}\rangle$. Also, $\lambda_1\in\langle\mathcal{G}\rangle$, and similarly $\lambda_2\in\langle\mathcal{G}_{\lambda_0}\rangle$, $\lambda_3\in\langle\mathcal{G}_{\lambda_0\lambda_1}\rangle$ etc. Since

$$\langle\mathcal{G}\rangle\geq\langle\mathcal{G}_{\lambda_0}\rangle\geq\ldots\geq\langle\mathcal{G}_{\lambda_0\ldots\lambda_n}\rangle,$$

$\lambda_2\ldots\lambda_n\in\langle\mathcal{G}_{\lambda_0}\rangle$, whence $1\neq h^{\boldsymbol{\lambda}}\in\langle\mathcal{G}_{\lambda_0}\rangle$. By Remark 2.2(2), $h^{\boldsymbol{\lambda}}m_{\mathcal{G}} = \lambda_0$. □

Before proceeding to the discussion of normal forms, a result for later use will be proved.

Lemma 3.1. *Let* $\mathcal{G} = (G,\{G_\lambda\mid\lambda\in\Lambda\},\{\leq_i\mid i\in\mathbb{N}\})$ *and*

$$\overline{\mathcal{G}} = (G,\left\{G_\lambda\mid\lambda\in\overline{\Lambda}\right\},\{\preceq_i\mid i\in\mathbb{N}\}).$$

be objects of F *with* $\mathcal{G}\leq\overline{\mathcal{G}}$. *Then every* $\mathcal{G}$*-descent sequence is a* $\overline{\mathcal{G}}$*-descent sequence.*

Proof. Let $\boldsymbol{\lambda} = (\lambda_0,\lambda_1,\ldots)$ be a $\mathcal{G}$-descent sequence. It follows by induction on n, using Lemma 2.5, that $(\lambda_0,\ldots,\lambda_n,1,1,\ldots)$ is a $\overline{\mathcal{G}}$-descent sequence and $\mathcal{G}_{\lambda_0\ldots\lambda_n}\leq\overline{\mathcal{G}}_{\lambda_0\ldots\lambda_n}$. Since $\boldsymbol{\lambda} = (\lambda_0,\ldots,\lambda_n,1,1,\ldots)$ for sufficiently large n, the lemma follows. □

If $\boldsymbol{\lambda} = (\lambda_0,\ldots,\lambda_n,1,1,\ldots)$ is a $\mathcal{G}$-descent sequence, λ_j will be denoted by $\boldsymbol{\lambda}(j)$. This is to allow for the finite sequence of descent sequences occurring in the normal form. The normal form is as follows.

Definition. If $\mathcal{G} = (G,\{G_\lambda\mid\lambda\in\Lambda\},\{\leq_i\mid i\in\mathbb{N}\})$ is an element of F and $g\in G$, then a $\mathcal{G}$*-normal form* for g is an expression $g = g_1^{\boldsymbol{\lambda}_1}\ldots g_k^{\boldsymbol{\lambda}_k}$ for some $\mathcal{G}$-descent sequences $\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_k$ with $\boldsymbol{\lambda}_1>\ldots>\boldsymbol{\lambda}_k$ in the lexicographic order, where $1\neq g_i\in G_{\boldsymbol{\lambda}_i(0)}$ for $1\leq i\leq k$.

If A is a set of $\mathcal{G}$-descent sequences and g has such an expression with all $\boldsymbol{\lambda}_i$ belonging to A, g is said to have a normal form with exponents from A.

Note that $k = 0$ is allowed, that is, 1_G always has a normal form, with exponents from any set of $\mathcal{G}$-descent sequences. The next lemma is needed to show that the normal form, if it exists, is unique.

Lemma 3.2. *Suppose* $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i | i \in \mathbb{N}\})$ *is an object of* F, $g \in G$ *and* g *has a normal form*

$$g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$$

where $k \geq 1$. *Let* l *be such that* $\boldsymbol{\lambda}_{l+1}(0) = \dots = \boldsymbol{\lambda}_k(0)$ *but* $\boldsymbol{\lambda}_{l+1}(0) <_0 \boldsymbol{\lambda}_l(0)$ *(*$l = 0$ *if* $\boldsymbol{\lambda}_1(0) = \dots = \boldsymbol{\lambda}_k(0)$*), so* $0 \leq l < k$. *Then*

(1) *if* $g \in G_\nu$ *for some* $\nu \in \Lambda$, *then* $k = 1$;
(2) $g \neq 1$;
(3) $gm_{\mathcal{G}} = \boldsymbol{\lambda}_k(0)$;
(4) $g' = g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l}$ *and* $g^* = g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k}$.

Proof. First, since $g_{l+1}^{\boldsymbol{\lambda}_{l+1}}, \dots, g_k^{\boldsymbol{\lambda}_k}$ all belong to $\langle \mathcal{G}_{\boldsymbol{\lambda}_k(0)} \rangle$ (cf the proof of Remark 3.1), $g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k} \in \langle \mathcal{G}_{\boldsymbol{\lambda}_k(0)} \rangle$. Also, by Remark 3.1, $g_i^{\boldsymbol{\lambda}_i} m_{\mathcal{G}} = \boldsymbol{\lambda}_i(0)$ for $1 \leq i \leq k$. For $1 \leq i \leq l$,

$$\boldsymbol{\lambda}_k(0) <_0 \boldsymbol{\lambda}_l(0) \leq_0 \boldsymbol{\lambda}_i(0)$$

hence $g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l} \in \ast_{\boldsymbol{\lambda}_k(0) <_0 \lambda} G_\lambda$. Since $g = (g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l})(g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k})$, (4) follows at once from (3).

Now suppose (1) is false, and take $\mathcal{G}$, an object of F, $\mathcal{G}$-descent sequences $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k$ and $g_i \in G_{\boldsymbol{\lambda}_i(0)} \setminus \{1\}$, where $k \geq 2$, such that $g := g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k} \in G_\nu$ for some $\nu \in \Lambda$, and with k as small as possible.

Let $p : G \to \ast_{\boldsymbol{\lambda}_k(0) <_0 \lambda} G_\lambda$ be the projection map. Then $gp \in G_\nu$, either because $\nu > \boldsymbol{\lambda}_k(0)$, so $gp = g$, or otherwise because $gp = 1$, and

$$gp = g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l}.$$

It follows that $g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k} \in G_\nu$, and since $l < k$, $l \leq 1$ by minimality of k. Suppose $l = 1$. Then $g_2^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k} \neq 1$ by minimality of k and because $g_k \neq 1$. Hence $\nu = (g_2^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k})m_{\mathcal{G}} = \boldsymbol{\lambda}_k(0)$ by Remark 2.2(2). Also, $g_1^{\boldsymbol{\lambda}_1} \in G_\nu$, so $\nu = g_1^{\boldsymbol{\lambda}_1} m_{\mathcal{G}} = \boldsymbol{\lambda}_1(0)$. But as noted above, $\boldsymbol{\lambda}_k(0) <_0 \boldsymbol{\lambda}_1(0)$, a contradiction. Hence $l = 0$, so $\boldsymbol{\lambda}_1(0) = \dots = \boldsymbol{\lambda}_k(0)$ and (from the first sentence of the proof) $g \in \langle \mathcal{G}_{\boldsymbol{\lambda}_k(0)} \rangle$. Hence $g \in G_{\lambda_k(0)}$, either because $g = 1$ or because $\nu = gm_{\mathcal{G}} = \lambda_k(0)$ by Remark 2.2(2).

For $1 \leq i \leq k$, let $h_i = g_i^{\boldsymbol{\lambda}_i(1)}$, so $h_i \in G_{\lambda_i(1)}$, and let $\boldsymbol{\lambda}'_i = (\boldsymbol{\lambda}_i(1), \boldsymbol{\lambda}_i(2), \dots)$. Then $\boldsymbol{\lambda}'_i$ is a $\mathcal{G}_{\lambda_k(0)}$-descent sequence, $\boldsymbol{\lambda}'_1 > \dots > \boldsymbol{\lambda}'_k$ in the lexicographic order and $h_i \neq 1$ for $1 \leq i \leq k$. Therefore g has a $\mathcal{G}_{\lambda_k(0)}$-normal form:

$$g = h_1^{\boldsymbol{\lambda}'_1} \dots h_k^{\boldsymbol{\lambda}'_k}.$$

Using the argument above, noting that $G_{\lambda_k(0)}$ is a free factor of $\mathcal{G}_{\lambda_k(0)}$, $\boldsymbol{\lambda}'_1(0) = \ldots = \boldsymbol{\lambda}'_k(0)$, that is, $\boldsymbol{\lambda}_1(1) = \ldots = \boldsymbol{\lambda}_k(1)$. Continuing, it follows by induction on n that

$$\boldsymbol{\lambda}_1(n) = \ldots = \boldsymbol{\lambda}_k(n)$$

for all $n \geq 0$, that is, $\boldsymbol{\lambda}_1 = \ldots = \boldsymbol{\lambda}_k$. Since $\boldsymbol{\lambda}_1 > \ldots > \boldsymbol{\lambda}_k$ and $k \geq 2$, this is a contradiction, establishing part (1).

Now (2) follows, since if $g = 1$, $k = 1$ by (1), hence $g = g_1^{\boldsymbol{\lambda}_1} \neq 1$ since $g_1 \neq 1$, a contradiction. It remains to prove (3).

By (2), $g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \ldots g_k^{\boldsymbol{\lambda}_k} \neq 1$, and by Remark 2.2(2), $(g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \ldots g_k^{\boldsymbol{\lambda}_k})m_{\mathcal{G}} = \boldsymbol{\lambda}_k(0)$. Since

$$g_1^{\boldsymbol{\lambda}_1} \ldots g_l^{\boldsymbol{\lambda}_l} \in \mathop{\ast}_{\boldsymbol{\lambda}_k(0) <_0 \lambda} G_\lambda,$$

it follows by Remark 2.2(1) that $gm_{\mathcal{G}} = \boldsymbol{\lambda}_k(0)$. □

The next lemma is an addendum to part (1) of Lemma 3.2.

Lemma 3.3. *Suppose $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ is an object of F and $\nu \in \Lambda$. If $g^{\boldsymbol{\lambda}} \in G_\nu$, where $\boldsymbol{\lambda}$ is a $\mathcal{G}$-descent sequence and $1 \neq g \in \boldsymbol{\lambda}(0)$, then $\nu = \boldsymbol{\lambda}(0)$ and $\boldsymbol{\lambda} = (\nu, 1, 1, \ldots)$.*

Proof. By Remark 3.1, $\nu = g^{\boldsymbol{\lambda}} m_{\mathcal{G}} = \boldsymbol{\lambda}(0)$. Suppose $\boldsymbol{\lambda}(1) \neq 1$. Then

$$g^{\boldsymbol{\lambda}} = (g^{\boldsymbol{\lambda}(1)})^h$$

where $h = \boldsymbol{\lambda}(2)\boldsymbol{\lambda}(3)\ldots$ (a finite product) and $h \in \langle \mathcal{G}_\nu \rangle$ (cf the proof of Remark 3.1). But $g^{\boldsymbol{\lambda}}$ and $g^{\boldsymbol{\lambda}(1)}$ are in different free factors of $\mathcal{G}_\nu$ (G_ν and $\boldsymbol{\lambda}(1)^{-1}G_\nu\boldsymbol{\lambda}(1)$ respectively), so are not conjugate in $\langle \mathcal{G}_\nu \rangle$ (see, for example, [4, Ch.4, Theorem 1.4]), a contradiction. Hence $\boldsymbol{\lambda}(1) = 1$. It follows that $G_{\boldsymbol{\lambda}(1)} = \boldsymbol{\lambda}(1)^{-1}G_\nu\boldsymbol{\lambda}(1) = G_\nu$, so $g \in G_{\boldsymbol{\lambda}(1)}$. Also, $\boldsymbol{\lambda}' := (\boldsymbol{\lambda}(1), \boldsymbol{\lambda}(2), \ldots)$ is a $\mathcal{G}_\nu$-descent sequence, and $g^{\boldsymbol{\lambda}'} = g^{\boldsymbol{\lambda}} \in G_{\boldsymbol{\lambda}(1)}$. Applying the argument above, $\boldsymbol{\lambda}'(1) = \boldsymbol{\lambda}(2) = 1$. Continuing (formally, by induction on n), $\boldsymbol{\lambda}(n) = 1$ for all $n \geq 1$, as required. □

Theorem 3.4. *Suppose $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ is an object of F and let $g \in G$. If g has a normal form*

$$g = g_1^{\boldsymbol{\lambda}_1} \ldots g_k^{\boldsymbol{\lambda}_k}$$

where $k \geq 0$, then k, $g_1, \ldots, g_k$ and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k$ are uniquely determined by g.

Proof. In view of Lemma 3.2(2), it can be assumed that $g \neq 1$ and $k \geq 1$. The proof will use induction on $\mathcal{G}$-length. If $g \in G_\nu$ for some $\nu \in \Lambda$, uniqueness of the expression in normal form follows from Lemmas 3.2(1) and 3.3, so assume $g \notin G_\nu$ for any $\nu \in \Lambda$.

Let l be such that $\boldsymbol{\lambda}_l(0) > \boldsymbol{\lambda}_{l+1}(0) = \ldots = \boldsymbol{\lambda}_k(0)$ (so $0 \leq l < k$), $l = 0$ meaning that $\boldsymbol{\lambda}_1(0) = \ldots = \boldsymbol{\lambda}_k(0)$). By Lemma 3.2, $\boldsymbol{\lambda}_k(0) = gm_{\mathcal{G}}$, which will be denoted by ν, and $g' = g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l}$ and $g^* = g_{l+1}^{\boldsymbol{\lambda}_{l+1}} \dots g_k^{\boldsymbol{\lambda}_k}$. This gives a $\mathcal{G}_\nu$-normal form for g^*, namely

$$g^* = h_{l+1}^{\boldsymbol{\lambda}'_{l+1}} \dots h_k^{\boldsymbol{\lambda}'_k},$$

where $h_i = g_i^{\boldsymbol{\lambda}_i(1)}$ and $\boldsymbol{\lambda}'_i = (\boldsymbol{\lambda}_i(1), \boldsymbol{\lambda}_i(2), \dots)$. Applying a similar argument to that just given, with g^* in place of g, shows that any $\mathcal{G}$-normal form for g^* gives a $\mathcal{G}_\nu$-normal form for g^* in the same way. Since g^* has shorter $\mathcal{G}_\nu$-length than the $\mathcal{G}$-length of g, by induction it has a unique $\mathcal{G}_\nu$-normal form, hence has a unique $\mathcal{G}$-normal form. If $g' = 1$, then this is the unique $\mathcal{G}$-normal form of g.

Suppose $g' \neq 1$. Then g' has shorter $\mathcal{G}$-length than g, so has a unique $\mathcal{G}$-normal form, and it follows by Lemma 3.2(4) that the $\mathcal{G}$-normal form of g is unique. □

Next, the question of existence of a normal form will be considered.

Definition. Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of F, and let A be a subset of Λ. For positive integers k, set

$$kA := \{g_1 \dots g_k \mid g_i \in G_{\lambda_i},\ \text{where } \lambda_i \in A \text{ for } 1 \leq i \leq k\}.$$

Note that $kA \subseteq (k+1)A$ ($g_i = 1$ is allowed).

Definition. Two elements g, h of $\langle \mathcal{G} \rangle$ are said to have compatible normal forms if they have normal forms, say $g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$, $h = h_1^{\boldsymbol{\mu}_1} \dots h_l^{\boldsymbol{\mu}_l}$ such that $\{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_l\}$ is totally ordered in the lexicographic order on $\mathcal{G}$-descent sequences.

Remark 3.2. If $\nu \in \Lambda$ and $\boldsymbol{\mu} = (\mu_0, \mu_1, \dots)$ is a $\mathcal{G}_\nu$-descent sequence, then

$$\boldsymbol{\lambda} := (\nu, \mu_0, \mu_1, \dots)$$

is a $\mathcal{G}$-descent sequence, and if $h \in G_\nu^{\mu_0}$ (the free factor of $\mathcal{G}_\nu$ corresponding to μ_0), so $h = g^{\mu_0}$ for some $g \in G_\nu$, then $h^{\boldsymbol{\mu}} = g^{\boldsymbol{\lambda}}$.

To prove existence of normal forms, an extra hypothesis is needed.

Definition. Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of F. Then $\mathcal{G}$ is said to satisfy CNC if the following condition is satisfied.

if two elements of G have compatible normal forms, then
they are $\leq_i$-comparable, for all $i \geq 1$.

The next proposition is the existence theorem for normal forms which will be used to construct total orders on free products in the next section. It is based on [3, Lemma 11].

Proposition 3.5. *Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of* F *which satisfies* CNC.

Let A be a subset of Λ, totally ordered by $\leq_0$ and let $k \geq 1$. Then there is a set B of $\mathcal{G}$-descent sequences, totally ordered by the lexicographic order, such that *every element of kA has a normal form with exponents from B. Consequently, kA is totally ordered by $\leq_i$, for all $i \geq 1$.*

Proof. First note that, if $\nu \in \Lambda$ and two elements of $\langle \mathcal{G}_\nu \rangle$ have compatible $\mathcal{G}_\nu$-normal forms, then they have compatible $\mathcal{G}$-normal forms. This follows easily from Remark 3.2. It then follows that $\mathcal{G}_\nu$ satisfies CNC.

The proof that the proposition is true for all such A is by induction on k. If $k = 1$, take

$$B = \{(\lambda, 1, 1, \dots) \mid \lambda \in A\}\,.$$

Assume that $k \geq 2$ and it is true for $k - 1$.

For $\nu \in A$, let $A_\nu := \{\mu \in A \mid \nu <_0 \mu\}$; then $(k-1)A_\nu \subseteq \Lambda_\nu$. By the induction hypothesis, $(k-1)A_\nu$ is totally ordered by $\leq_i$ for all $i \geq 1$, in particular for $i = 1$, so it is a totally ordered subset of Λ_ν. By the induction hypothesis, there is a set of $\mathcal{G}_\nu$-descent sequences C_ν, totally ordered by the lexicographic order, such that every element of

$$D_\nu := \left\{ g_1^{a_1} \dots g_{k-1}^{a_{k-1}} \mid g_i \in G_\nu,\ a_i \in (k-1)A_\nu \right\}$$

has a $\mathcal{G}_\nu$-normal form with exponents from C_ν. Let

$$B_\nu := \{(\nu, \mu_0, \mu_1, \dots) \mid (\mu_0, \mu_1, \dots) \in C_\nu\}\,.$$

Then B_ν is a set of $\mathcal{G}$-descent sequences, totally ordered by the lexicographic order, and every element of D_ν has a $\mathcal{G}$-normal form with exponents from B_ν, by Remark 3.2. Let $B := \bigcup_{\nu \in A} B_\nu$. Since A is totally ordered by $\leq_0$, B is totally ordered by the lexicographic order. In fact, if ν, $\nu' \in A$ and $\nu <_0 \nu'$, then for all $\boldsymbol{\lambda} \in B_\nu$ and $\boldsymbol{\lambda}' \in B_{\nu'}$, $\boldsymbol{\lambda} < \boldsymbol{\lambda}'$ in the lexicographic order.

Let $g \in kA$; it will be shown that g has a $\mathcal{G}$-normal form with exponents from B. This is obvious if $g = 1$, so assume $g \neq 1$ and write g as $g = g_1 \dots g_k$, where $g_i \in G_{\lambda_i}$ and $\lambda_i \in A$ for $1 \leq i \leq k$. Since A is totally ordered by $\leq_0$, $gm_{\mathcal{G}}$ is defined, and will be denoted by ν. The $\mathcal{G}$-reduced form of g has length at most k, and writing $g = g'g^*$ using equation (2.1), it is easy to see that $g^* \in D_\nu$.

If $g' \neq 1$ then $\nu <_0 g'm_{\mathcal{G}}$, and repeating this procedure with g' in place of g, and continuing, eventually g can be written as

$$g = r_1 \dots r_p$$

where $r_i \in D_{\nu_i}$ for some $\nu_i \in A$ such that $\nu_p <_0 \dots <_0 \nu_1$. Now r_i has a normal form with exponents from B_{ν_i}, and substituting these expressions for r_i in the expression for g shows that g has a $\mathcal{G}$-normal form with exponents from B. □

Corollary 3.6. *Suppose $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ is an object of F_0. Then every element g of G has a $\mathcal{G}$-normal form.*

Proof. Obviously $\mathcal{G}$ satisfies CNC, and one can take $A = \Lambda$ in Proposition 3.5. Since every element of G belongs to $k\Lambda$ for some k, the corollary follows. □

The next result in this section shows that, if an element $g \neq 1$ in $\langle \mathcal{G} \rangle$ has a normal form, then $g \in X_{\mathcal{G}} \cup X_{\mathcal{G}}^{-1}$.

Proposition 3.7. *Suppose* $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ *is an object of* F*, let* $g \in G \setminus \{1\}$ *and assume* g *has a normal form* $g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$ *with* $k \geq 1$*. Then*

(1) *if* $g_1 > 1$ *in* $G_{\boldsymbol{\lambda}_1(0)}$ *then* $g \in X_{\mathcal{G}}$*;*
(2) *if* $g_1 < 1$ *in* $G_{\boldsymbol{\lambda}_1(0)}$ *then* $g^{-1} \in X_{\mathcal{G}}$*.*

Proof. The proof is by induction on $\mathcal{G}$-length. If $g \in G_\nu$ for some ν then $g = g_1$, by Lemmas 3.2(1) and 3.3, and the proposition follows in this case. Suppose $g \notin G_\nu$, for any $\nu \in \Lambda$. By Lemma 3.2, g decomposes as $g = g'g^*$, where $g' \in \Lambda_\nu$, $g^* \in \langle \mathcal{G}_\nu \rangle$ and $\nu = gm_{\mathcal{G}}$, moreover $g' = g_1^{\boldsymbol{\lambda}_1} \dots g_l^{\boldsymbol{\lambda}_l}$ for some $l \geq 0$. Further, $g^{-1}m_{\mathcal{G}} = gm_{\mathcal{G}}$ and the corresponding decomposition of g^{-1} is given by equations 2.2.

If $g' \neq 1$ then $l \geq 1$, and g' has shorter $\mathcal{G}$-length than g, so by induction and the definition of $X_{\mathcal{G}}$,

$$g_1 > 1 \text{ in } G_{\boldsymbol{\lambda}_1(0)} \Rightarrow g' \in X_{\mathcal{G}} \Rightarrow g \in X_{\mathcal{G}}$$

and

$$g_1 < 1 \text{ in } G_{\boldsymbol{\lambda}_1(0)} \Rightarrow (g')^{-1} \in X_{\mathcal{G}} \Rightarrow g^{-1} \in X_{\mathcal{G}}.$$

Suppose $g' = 1$. Then $l = 0$ by Lemma 3.2(2), and by definition of l, $\boldsymbol{\lambda}_1(0) = \dots \boldsymbol{\lambda}_k(0)$; denote $\boldsymbol{\lambda}_1(0)$ by ν. Now g can be written as

$$g = g^* = h_1^{\boldsymbol{\lambda}'_1} \dots h_k^{\boldsymbol{\lambda}'_k}$$

where $h_i = g_i^{\boldsymbol{\lambda}_i(1)}$ and $\boldsymbol{\lambda}'_i = (\boldsymbol{\lambda}_i(1), \boldsymbol{\lambda}_i(2), \dots)$. This is the normal form of g^* in $\mathcal{G}_\nu$, and the $\mathcal{G}_\nu$-length of g^* is less than the $\mathcal{G}$-length of $g = g^*$. Again by induction and the definition of $X_{\mathcal{G}}$,

$$h_1 > 1 \text{ in } G_{\boldsymbol{\lambda}'_1(0)} \Rightarrow g^* \in X_{\mathcal{G}_\nu} \Rightarrow g \in X_{\mathcal{G}}$$

and

$$h_1 < 1 \text{ in } G_{\boldsymbol{\lambda}'_1(0)} \Rightarrow (g^*)^{-1} = (g^{-1})^* \in X_{\mathcal{G}_\nu} \Rightarrow g^{-1} \in X_{\mathcal{G}}.$$

Also, $G_{\boldsymbol{\lambda}'_1(0)} = G_{\boldsymbol{\lambda}_1(1)} = \boldsymbol{\lambda}_1(1)^{-1} G_\nu \boldsymbol{\lambda}_1(1)$; it follows by definition of $\mathcal{G}_\nu$ that

$$h_1 > 1 \text{ in } G_{\boldsymbol{\lambda}'_1(0)} \Leftrightarrow g_1 > 1 \text{ in } G_\nu \text{ and } h_1 < 1 \text{ in } G_{\boldsymbol{\lambda}'_1(0)} \Leftrightarrow g_1 < 1 \text{ in } G_\nu,$$

completing the proof. □

Corollary 3.8. *Suppose* $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ *is an object of* F*. Let*

$$\begin{aligned} S := &\left\{ g^{\boldsymbol{\lambda}} h^{\boldsymbol{\mu}} \mid \boldsymbol{\lambda}, \boldsymbol{\mu} \text{ are } \mathcal{G}\text{-descent sequences}, \boldsymbol{\lambda} > \boldsymbol{\mu}, g \in G_{\boldsymbol{\lambda}(0)}, h \in G_{\boldsymbol{\mu}(0)}, \right. \\ &\quad \left. g > 1 \text{ in } G_{\boldsymbol{\lambda}(0)} \right\} \\ &\quad \cup \left\{ g^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \text{ is a } \mathcal{G}\text{-descent sequence}, g \in G_{\boldsymbol{\lambda}(0)}, g > 1 \text{ in } G_{\boldsymbol{\lambda}(0)} \right\}. \end{aligned}$$

Then $S \subseteq X_{\mathcal{G}}$, and if $\mathcal{G}$ is an object of $\mathbf{F}_0$, then $P_{\mathcal{G}}$ is the subsemigroup of G generated by S.

Proof. The elements of S belong to $X_{\mathcal{G}}$ by Proposition 3.7. Suppose $\mathcal{G}$ is an object of $\mathbf{F}_0$, and let P be the subsemigroup generated by S. It follows that $P \subseteq P_{\mathcal{G}}$. By Corollary 3.6, every element of g has a normal form, and by Proposition 2.4, $P_{\mathcal{G}} = X_{\mathcal{G}}$. By Proposition 3.7, if $g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$ $(k \geq 1)$ in normal form, then $g \in P_{\mathcal{G}}$ if and only if $g_1 > 1$ in $G_{\boldsymbol{\lambda}_1(0)}$. Thus it suffices to show that, if $g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$ $(k \geq 1)$ in normal form and $g_1 > 1$ in $G_{\boldsymbol{\lambda}_1(0)}$, then $g \in P$. This will be accomplished by induction on k.

If $k \leq 2$ then $g \in S$, so assume $k \geq 3$. Suppose $g_{k-1} > 1$. Then

$$g = (g_1^{\boldsymbol{\lambda}_1} \dots g_{k-2}^{\boldsymbol{\lambda}_{k-2}})(g_{k-1}^{\boldsymbol{\lambda}_{k-1}} g_k^{\boldsymbol{\lambda}_k}).$$

By induction, the first factor in this product is in P, and the second is in S, so $g \in P$.

Suppose $g_{k-1} < 1$. Then

$$g = (g_1^{\boldsymbol{\lambda}_1} \dots (g_{k-1}^2)^{\boldsymbol{\lambda}_{k-1}})((g_{k-1}^{-1})^{\boldsymbol{\lambda}_{k-1}} g_k^{\boldsymbol{\lambda}_k}).$$

The two factors are in normal form, and again the first factor is in P by induction, while the second is in S, hence $g \in P$, completing the proof. □

In the corollary, for any object $\mathcal{G}$ of $\mathbf{F}$, if P is the normal subsemigroup of G generated by S, it follows that $P \subseteq P_{\mathcal{G}}$. If the procedure of [3] were followed exactly, one would define $P_{\mathcal{G}}$ to be P, rather than use the recursive definition in §2. However, it is not clear that $P = P_{\mathcal{G}}$ in general.

Finally, here are two remarks which will be useful in the next section.

Remark 3.3. Let $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$ be an object of $\mathbf{F}$ and assume all G_λ are non-trivial. Suppose $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ and take $g_\lambda \in G_\lambda$, $g_\mu \in G_\mu$ with $g_\lambda > 1$ and $g_\mu < 1$. Put $g = g_\lambda g_\mu$. Then:
if $\mu <_0 \lambda$ then $g' = g_\lambda$, so $g \in X_{\mathcal{G}}$;
if $\lambda <_0 \mu$ then $g = g_\mu(g_\mu^{-1} g_\lambda g_\mu)$, so $g' = g_\mu$, hence $g \notin X_{\mathcal{G}}$.

It follows that, if Λ has at least two elements, and $\mathcal{G}'$ is an object of $\mathbf{F}$, obtained from $\mathcal{G}$ by changing the total order $\leq_0$, and possibly changing $\{\leq_i \mid i \in \mathbb{N}_{>0}\}$, but otherwise leaving $\mathcal{G}$ unchanged, then $X_{\mathcal{G}} \neq X_{\mathcal{G}'}$.

Remark 3.4. If $\mathcal{G}$ is an object of $\mathbf{F}$ and $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots)$ is a $\mathcal{G}$-descent sequence, then

$$\langle \mathcal{G}_{\lambda_0 \dots \lambda_n} \rangle \cap X_{\mathcal{G}} = X_{\mathcal{G}_{\lambda_0 \dots \lambda_n}}$$

for $n \geq 0$. By induction on n, it suffices to show that $\langle \mathcal{G}_{\lambda_0} \rangle \cap X_{\mathcal{G}} = X_{\mathcal{G}_{\lambda_0}}$. But if $g \in \langle \mathcal{G}_{\lambda_0} \rangle$ and $g \neq 1$, then $gm_{\mathcal{G}} = \lambda_0$ by Remark 2.2(2), hence $g' = 1$ and $g = g^*$, so by definition $g \in X_{\mathcal{G}}$ if and only if $g \in X_{\mathcal{G}_{\lambda_0}}$.

4. Constructing Orders on Free Products

The objects of the category $\mathbf{G}$ defined in [2] are pairs $(G, \{G_\lambda \mid \lambda \in \Lambda\})$, where, for all $\lambda \in \Lambda$, G_λ is a totally ordered subgroup of G, the index set Λ is *totally* ordered and $G = \ast_{\lambda\in\Lambda} G_\lambda$. Given such an object, and a sequence whose terms are ± 1, an order will be constructed on G.

Fix an object $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\})$ of $\mathbf{G}$, and a sequence $s = (s_1, s_2, \ldots)$ with $s_i = \pm 1$ for all $i \geq 1$. Define

$$\mathcal{G}(0) := (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i \mid i \in \mathbb{N}\})$$

where $\leq_0$ is the given order on Λ and for $i \geq 1$, $\leq_i$ is the trivial partial order of equality on G; thus $\mathcal{G}(0)$ is an object of $\mathbf{F}$. Let $\leq$ be a total order on G; then $\leq$ extends $\leq_i$ for all $i \geq 1$, so by Corollary 2.6, $P_{\mathcal{G}(0)}$ is the positive cone for a partial order on G, which has an extension to a total order on G.

For $m \in \mathbb{N}$, assume an object $\mathcal{G}(m)$ of $\mathbf{F}$ has been defined with $\langle \mathcal{G}(m) \rangle = G$, such that $P_{\mathcal{G}(m)}$ is the positive cone for a partial order on G, which has an extension to a total order on G. Set

$$\mathcal{G}(m+1) := (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i^m \mid i \in \mathbb{N}\})$$

where $\leq_0^m$ is the given order on Λ, and for $i \geq 1$, $\leq_i^m$ is the partial order corresponding to $P_{\mathcal{G}(m)}$ if $s_i = 1$, and the reverse of this partial order if $s_i = -1$. Then $P_{\mathcal{G}(m+1)}$ is the positive cone of a partial order on G which has an extension to a total order on G, by Corollary 2.6. This recursively defines $\mathcal{G}(m)$ for all $m \geq 0$; $P_{\mathcal{G}(m)}$ will be abbreviated to P_m, and the corresponding partial order on G will be denoted by $\leq^m$.

Lemma 4.1. *For all $m \geq 0$, $\mathcal{G}(m) \leq \mathcal{G}(m+1)$ and $P_m \subseteq P_{m+1}$.*

Proof. Clearly $\mathcal{G}(0) \leq \mathcal{G}(1)$, hence $P_0 \subseteq P_1$ by Lemma 2.5, and it follows easily by induction on m that $\mathcal{G}(m) \leq \mathcal{G}(m+1)$ and $P_m \subseteq P_{m+1}$ for all m, using Lemma 2.5. □

Let $P_\omega := \bigcup_{m=0}^{\infty} P_m$. It follows easily from Lemma 4.1 that P_ω is the positive cone of a partial order on G, which will be denoted by $\leq^\omega$. This gives an object of $\mathbf{F}$,

$$\mathcal{G}(\omega) := (G, \{G_\lambda \mid \lambda \in \Lambda\}, \{\leq_i^\omega \mid i \in \mathbb{N}\}),$$

where again $\leq_0^\omega$ is the given order on Λ and for $i \geq 1$, $\leq_i^\omega$ is $\leq^\omega$ if $s_i = 1$, and the reverse order to $\leq^\omega$ if $s_i = -1$. It will be shown that P_ω defines a total order on G, so that, in fact, $\mathcal{G}(\omega)$ is an object of $\mathbf{F}_0$.

If $(\lambda_0, \lambda_1 \ldots)$ is a $\mathcal{G}(m)$-descent sequence, denote the index set of $(\mathcal{G}(m))_{\lambda_0\ldots\lambda_n}$ by

$$\Lambda_{\lambda_0\ldots\lambda_n}(m)$$

for $0 \leq m \leq \omega$. By equation 3.1, $\Lambda_{\lambda_0\ldots\lambda_n}(m) = \ast_{\lambda_n <_n^m \lambda} G_{\lambda_0}^{\lambda_1\ldots\lambda_{n-1}\lambda}$, where $\lambda \in \Lambda_{\lambda_0\ldots\lambda_{n-1}}(m)$.

Lemma 4.2. *If $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots)$ is a $\mathcal{G}(\omega)$-descent sequence, then there exists $M \in \mathbb{N}$ such that $\boldsymbol{\lambda}$ is a $\mathcal{G}(m)$-descent sequence for all $m \geq M$.*

Proof. First, it will be shown by induction on n that for all such $\mathcal{G}$-descent sequences $\boldsymbol{\lambda}$, $(\lambda_0, \ldots, \lambda_n, 1, 1, \ldots)$ is a $\mathcal{G}(m)$-descent sequence for some $m \in \mathbb{N}$. This is clear for $n = 0$; assume it is true for n, and suppose that $(\lambda_0, \ldots, \lambda_n, 1, 1, \ldots)$ is a $\mathcal{G}(m)$-descent sequence. Then

$$\lambda_{n+1} \in \Lambda_{\lambda_0 \ldots \lambda_n}(\omega) = \mathop{\ast}_{\lambda_n <_n^\omega \lambda} G_{\lambda_0}^{\lambda_1 \ldots \lambda_{n-1} \lambda}$$

where $\lambda \in \Lambda_{\lambda_0 \ldots \lambda_{n-1}}(\omega)$. so λ_{n+1} can be written as $\lambda_{n+1} = \alpha_1 \ldots \alpha_k$, where $\alpha_i \in G_{\lambda_0}^{\lambda_1 \ldots \lambda_{n-1} \mu_i}$ for some μ_i such that $\lambda_n <_n^\omega \mu_i$ and $\mu_i \in \Lambda_{\lambda_0 \ldots \lambda_{n-1}}(\omega)$, so $(\lambda_0, \ldots, \lambda_{n-1}, \mu_i, 1, 1 \ldots)$ is a $\mathcal{G}(\omega)$-descent sequence. By the induction hypothesis, $(\lambda_0, \ldots, \lambda_{n-1}, \mu_i, 1, 1 \ldots)$ is a $\mathcal{G}(m_i)$-descent sequence for some $m_i \in \mathbb{N}$. Also, since $P_\omega = \bigcup_{i=0}^\infty P_i$, it follows that $\lambda_n <_n^{r_i} \mu_i$ for some $r_i \in \mathbb{N}$. Let

$$r := \max \{m_1, \ldots, m_k, r_1, \ldots, r_k, m\}.$$

By Lemma 3.1, $(\lambda_0, \ldots, \lambda_n, 1, 1, \ldots)$ is a $\mathcal{G}(r)$-descent sequence, and for $1 \leq i \leq k$, so is $(\lambda_0, \ldots, \lambda_{n-1}, \mu_i, 1, 1, \ldots)$, hence $\mu_i \in \Lambda_{\lambda_0 \ldots \lambda_{n-1}}(r)$. Since $P_{r_i} \subseteq P_r$, $\lambda_n <_n^r \mu_i$ for $1 \leq i \leq k$. Therefore,

$$\begin{aligned} \lambda_{n+1} &\in \mathop{\ast}_{\lambda_n <_n^r \lambda} G_{\lambda_0}^{\lambda_1 \ldots \lambda_{n-1} \lambda} \quad (\lambda \in \Lambda_{\lambda_0 \ldots \lambda_{n-1}}(r)) \\ &= \Lambda_{\lambda_0 \ldots \lambda_n}(r). \end{aligned}$$

It follows that $(\lambda_0, \ldots, \lambda_{n+1}, 1, 1, \ldots)$ is a $\mathcal{G}(r)$-descent sequence, completing the inductive proof.

Now choose n large enough so that $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n, 1, 1, \ldots)$, and $M \in \mathbb{N}$ such that this is a $\mathcal{G}(M)$-descent sequence. Then by Lemma 3.1, $\boldsymbol{\lambda}$ is a $\mathcal{G}(m)$-descent sequence for all $m \geq M$. □

Corollary 4.3. *If $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are $\mathcal{G}(\omega)$-descent sequences and $\boldsymbol{\lambda} < \boldsymbol{\mu}$ in the lexicographic order on $\mathcal{G}(\omega)$-descent sequences, then there exists $M \in \mathbb{N}$ such that, for all m with $m \geq M$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are $\mathcal{G}(m)$-descent sequences and $\boldsymbol{\lambda} < \boldsymbol{\mu}$ in the lexicographic order on $\mathcal{G}(m)$-descent sequences.*

Proof. This follows easily from Lemma 4.2 and the fact that $P_\omega = \bigcup_{m=0}^\infty P_m$, with $P_m \subseteq P_{m+1}$ for $m \in \mathbb{N}$. □

Lemma 4.4. *Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n$ be $\mathcal{G}(m)$-descent sequences, with $m \in \mathbb{N}$, $n \geq 1$, $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$ totally ordered, and $\boldsymbol{\lambda} > \max \{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$ (in the lexicographic order). Let $g \in G_{\boldsymbol{\lambda}(0)}$ with $g > 1$ and let $g_i \in G_{\boldsymbol{\lambda}_i(0)}$ for $1 \leq i \leq n$. Then*

$$g_1^{\boldsymbol{\lambda}_1} \ldots g_n^{\boldsymbol{\lambda}_n} <^{m+1} g^{\boldsymbol{\lambda}}.$$

Proof. It follows from Corollary 3.8 that $\{g_1^{\boldsymbol{\lambda}_1}, \dots, g_n^{\boldsymbol{\lambda}_n}\}$ is totally ordered by $\leq^{m+1}$. Choose i so that $g_i^{\boldsymbol{\lambda}_i}$ is the largest element of this set. Then

$$g_1^{\boldsymbol{\lambda}_1} \dots g_n^{\boldsymbol{\lambda}_n} \leq^{m+1} (g_i^n)^{\boldsymbol{\lambda}_i}$$

and again by Corollary 3.8, $(g_i^n)^{\boldsymbol{\lambda}_i} <^{m+1} g^{\boldsymbol{\lambda}}$. □

Lemma 4.5. *If two elements of G have compatible $\mathcal{G}(\omega)$-normal forms, then they are $\leq^\omega$- comparable.*

Proof. Let g, h have compatible $\mathcal{G}(\omega)$-normal forms, say

$$g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}, \ h = h_1^{\boldsymbol{\mu}_1} \dots h_l^{\boldsymbol{\mu}_l}$$

where $\{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_l\}$ is totally ordered. It will be shown by induction on $k+l$ that g, h are $\leq^\omega$-comparable. This is clear if $k+l=0$, so assume $k+l>0$. If $\boldsymbol{\lambda}_1 = \boldsymbol{\mu}_1$, let $g' = (h_1^{-1}g_1)^{\boldsymbol{\lambda}_1} g_2^{\boldsymbol{\lambda}_2} \dots g_k^{\boldsymbol{\lambda}_k}$, $h' = h_2^{\boldsymbol{\mu}_2} \dots h_l^{\boldsymbol{\mu}_l}$; then by induction g', h' are $\leq^\omega$-comparable, hence so are $g = h_1^{\boldsymbol{\mu}_1} g'$, $h = h_1^{\boldsymbol{\mu}_1} h'$.

Therefore, without loss of generality, assume $\boldsymbol{\lambda}_1 > \boldsymbol{\mu}_1$, or $k \geq 1$ and $l = 0$. By Lemma 4.2 and Corollary 4.3, there exists $m \in \mathbb{N}$ such that $g = g_1^{\boldsymbol{\lambda}_1} \dots g_k^{\boldsymbol{\lambda}_k}$, $h = h_1^{\boldsymbol{\mu}_1} \dots h_l^{\boldsymbol{\mu}_l}$ are $\mathcal{G}(m)$-normal forms.

Assume $g_1 > 1$. Then by Lemma 4.4, (or by Corollary 3.8 if $l=0$, $k=1$)

$$h_1^{\boldsymbol{\mu}_1} \dots h_l^{\boldsymbol{\mu}_l} (g_k^{-1})^{\boldsymbol{\lambda}_k} \dots (g_2^{-1})^{\boldsymbol{\lambda}_2} <^{m+1} g_1^{\boldsymbol{\lambda}_1}$$

hence $h <^{m+1} g$, so $h <^\omega g$ since $P_{m+1} \subseteq P_\omega$.

If $g_1 < 1$, then similarly

$$g_2^{\boldsymbol{\lambda}_2} \dots g_k^{\boldsymbol{\lambda}_k} (h_l^{-1})^{\boldsymbol{\mu}_l} \dots (h_1^{-1})^{\boldsymbol{\mu}_1} <^{m+1} (g_1^{-1})^{\boldsymbol{\lambda}_1}$$

and it follows that $g <^{m+1} h$, so $g <^\omega h$. □

Theorem 4.6. *The order $\leq^\omega$ is a total order on G.*

Proof. By Lemma 4.5, the hypotheses of Proposition 3.5 apply to $\mathcal{G}(\omega)$; take A to be the set of $\mathcal{G}(\omega)$-descent sequences

$$\{(\lambda, 1, 1, \dots) \mid \lambda \in \Lambda\}.$$

This is totally ordered by the lexicographic order, because Λ is totally ordered, being the original index set of the object $\mathcal{G}$ of $\mathbf{G}$. Then

$$kA = \{g_1 \dots g_k \mid \text{ for } 1 \leq i \leq k, \ g_i \in G_{\lambda_i} \text{ for some } \lambda_i \in \Lambda\}$$

so any two elements of G are in kA for sufficiently large k, and by Proposition 3.5, kA is totally ordered by $\leq^\omega$. Hence $\leq^\omega$ is a total order. □

Thus $\mathcal{G}(\omega)$ is an object of $\mathbf{F}_0$, hence $P_{\mathcal{G}(\omega)}$ is the positive cone for a total order on G, by Lemma 2.4. One could try to continue the construction of P_m for ordinals $m > \omega$ by defining $P_{\omega+1} = P_{\mathcal{G}(\omega)}$, but the next corollary shows that this gives nothing new.

Corollary 4.7. $P_\omega = P_{\mathcal{G}(\omega)}$.

Proof. Clearly $\mathcal{G}(m) \leq \mathcal{G}(\omega)$ for all $m \in \mathbb{N}$, so $P_m \subseteq P_{\mathcal{G}(\omega)}$ for all $m \in \mathbb{N}$ by Lemma 2.5, hence $P_\omega \subseteq P_{\mathcal{G}(\omega)}$. Since both P_ω and $P_{\mathcal{G}(\omega)}$ are positive cones for total orders on G, it follows that they are equal. $\square$

Thus, given the object $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\})$ of the category G, an order $\leq^\omega$ has been constructed on G. (As it stands, this does not give a proof of Vinogradov's Theorem, since it was necessary to use it in order to show that P_0 is the positive cone for a partial order on G. Even if an independent proof could be given, it would result in a very elaborate proof of this theorem.) The construction also depends on the sequence s, and to reflect this P_ω will now be denoted by P_s, and $\mathcal{G}(\omega)$ by $\mathcal{G}(s)$.

Proposition 4.8. *Suppose $\mathcal{G} = (G, \{G_\lambda \mid \lambda \in \Lambda\})$ is an object of the category G, Λ has at least two elements and all G_λ are non-trivial. If $s = (s_1, s_2, \ldots)$ and $s' = (s'_1, s'_2 \ldots)$ are sequences with s_i, $s'_i = \pm 1$ and $s \neq s'$, then $P_s \neq P_{s'}$.*

Proof. Suppose $s_n \neq s'_n$ with n as small as possible subject to this. Choose $\lambda_0 \in \Lambda$ such that $\lambda_0 < \mu$ for some $\mu \in \Lambda$. Then $\mathcal{G}(s)_{\lambda_0}$ satisfies the same hypotheses as $\mathcal{G}(s)$ (index set has at least two elements, all free factors non-trivial). Proceeding inductively, there is a $\mathcal{G}(s)$-descent sequence $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots \lambda_{n-1}, 1, \ldots)$, such that $\mathcal{G}_{\lambda_0 \ldots \lambda_{n-1}}(s)$ satisfies the same hypotheses as $\mathcal{G}(s)$. Then $\boldsymbol{\lambda}$ is also a $\mathcal{G}(s')$-descent sequence, and $\mathcal{G}_{\lambda_0 \ldots \lambda_{n-1}}(s)$ and $\mathcal{G}_{\lambda_0 \ldots \lambda_{n-1}}(s')$ have the same index set and corresponding free factors (as ordered groups), but with different orderings on the index set. (The two different orderings are reverse orderings.) By Remark 3.3, $X_{\mathcal{G}_{\lambda_0 \ldots \lambda_{n-1}}(s)} \neq X_{\mathcal{G}_{\lambda_0 \ldots \lambda_{n-1}}(s')}$, and by Remark 3.4, $X_{\mathcal{G}(s)} \neq X_{\mathcal{G}(s')}$. But by Theorem 4.6, $\mathcal{G}(s)$ is an object of F_0, so by Proposition 2.4 and Corollary 4.7, $P_s = P_{\mathcal{G}(s)} = X_{\mathcal{G}(s)}$, and similarly $P_{s'} = X_{\mathcal{G}(s')}$, hence $P_s \neq P_{s'}$. $\square$

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