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Embedding theorems for tree-free groups

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Abstract

We establish two embedding theorems for tree-free groups. The first result embeds a group G acting freely and without inversions on a Λ-tree $\mathbf{X}$ into a group $\widehat{\mathbf{G}}$ acting freely, without inversions, and transitively on a Λ-tree $\widehat{\mathbf{X}}$ in such a way that $\mathbf{X}$ embeds into $\widehat{\mathbf{X}}$ by means of a G-equivariant isometry. The second result embeds a group G acting freely and transitively on an $\mathbb{R}$-tree $\mathbf{X}$ into $\mathcal{RF}(H)$ for some suitable group H, again in such a way that $\mathbf{X}$ embeds G-equivariantly into the $\mathbb{R}$-tree $\mathbf{X}_H$ associated with $\mathcal{RF}(H)$. The group $\mathcal{RF}(H)$ referred to here belongs to a class of groups introduced and studied by the present authors in [**3**]. As a consequence of these two theorems, we find that $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees are in fact *universal* for free $\mathbb{R}$-tree actions. Moreover, our first embedding theorem throws light on the question, arising from the results of [**7**], whether a group endowed with a Lyndon length function L can always be embedded in a length-preserving way into a group with a *regular* Lyndon length function; modulo an obvious necessary restriction we show that this is indeed the case if L is free.

1. *Introduction*

By a tree-free group we mean a group having an action on a Λ-tree, for some (totally) ordered abelian group Λ, which is free and without inversions. This means that every non-identity element acts as a hyperbolic isometry (see [**2**, Chapter 3]). This paper arose from our efforts to verify some of the results in the paper of Alperin and Moss [**1**], one of the seminal papers on $\mathbb{R}$-trees. The first is Theorem 3.4, that a group G with a real length function can be embedded in a group $\hat{G}$ which is a "principal ideal group". The construction involves taking the corresponding $\mathbb{R}$-tree $\mathbf{X}$ on which G acts, and adding copies of $\mathbf{X}$, joined together at a single point, at every point in the closure of the set of branch points, then iterating the construction, to obtain an $\mathbb{R}$-tree $\mathbf{X}_\infty$. Elements of $\hat{G}$ are then defined as certain geodesic paths in $\mathbf{X}_\infty$. There is a natural length function on $\hat{G}$; however, the detailed proof that $\hat{G}$ is a group, and that $\mathbf{X}_\infty$ is the canonical tree associated to the length function, are omitted, and appear to be rather problematic in that context. Consequently, we have adopted a different approach, defining $\hat{G}$ as a set of equivalence classes of sequences under certain rewrite rules, defining the length function directly, then using the canonical construction of a tree from a length function to obtain an analogue of $\mathbf{X}_\infty$ (now called $\widehat{\mathbf{X}}$). We also make two changes; firstly, copies of $\mathbf{X}$ can be attached at any point, not just at points in the closure of the set of branch points. Secondly, when this is done, there is no need to confine attention to $\mathbb{R}$-trees;

the argument applies to Λ-trees, for any Λ. The outcome is the following improved result; cf. Theorem 5.4.

THEOREM. *Let G be a group acting freely and without inversions on a Λ-tree $\mathbf{X}$. Then there exist a group $\hat{G}$ and a Λ-tree $\widehat{\mathbf{X}}$, such that $\hat{G}$ acts freely, without inversions, and transitively on $\widehat{\mathbf{X}}$, together with an embedding $G \to \hat{G}$ and an equivariant isometry $\mathbf{X} \to \widehat{\mathbf{X}}$.*

Our argument bears no obvious resemblance to that in [**1**]; however, like the one in [**1**], it only applies to actions which are free and without inversions.

Our second embedding result is based on [**1**, Theorem 4.2]. Here, it is left unexplained why the function F_g in the proof is admissible. Specifically, this involves a certain quotient set S of a group G; it appears difficult to see how the map with domain S needed to define admissibility on p. 65 can be chosen so that it is injective. However, with some effort, the original argument can be adapted to show that, if a group G acts freely on an $\mathbb{R}$-tree $\mathbf{X}$, then there is an embedding of G into $\mathcal{RF}(H)$ for some group H, where $\mathcal{RF}(H)$ is one of the groups constructed and investigated in [**3**]. Moreover, there is an equivariant embedding of $\mathbf{X}$ into the canonical $\mathbb{R}$-tree $\mathbf{X}_H$ on which $\mathcal{RF}(H)$ acts.

To prove this, we let H be the quotient set S above, given an arbitrary group structure, and replace the ordered pairs (s_1, s_2) of elements of S used in [**1**] by $s_1^{-1}s_2$. We give the details to show that this construction works.

It is clear that our results depend heavily on the connection between Lyndon length functions and actions on Λ-trees, which we summarise next. The definition of a Lyndon length function is given after [**2**, Chapter 2, Corollary 4.5]. In particular, if $L : G \to \Lambda$ is a map, where G is a group and Λ is an ordered abelian group, we let

$$c(g, h) = \tfrac{1}{2}\big\{L(g) + L(h) - L(g^{-1}h)\big\}.$$

THEOREM 1.1. *Let G be a group and let Λ be an ordered abelian group.*

(i) *If G acts on a Λ-tree (X, d) and $x \in X$ is any point, define $L_x\colon G \to \Lambda$ by $L_x(g) = d(x, gx)$. Then L_x is a Lyndon length function satisfying $c(g, h) \in \Lambda$ for all $g, h \in G$.*

(ii) *Conversely, given a Lyndon length function $L\colon G \to \Lambda$ satisfying $c(g, h) \in \Lambda$ for all $g, h \in G$, there are a Λ-tree (X, d) on which G acts and a point $x \in X$ such that $L = L_x$.*

Proof. See [**2**, Chapter 2, Theorem 4.6] and the preceding paragraph.

The proof of (i) depends on the fact that Λ-trees are 0-hyperbolic. If (X, d) is any Λ-metric space, and $x \in X$, set

$$(y \cdot z)_x = \tfrac{1}{2}\big\{d(y, x) + d(z, x) - d(x, y)\big\}, \qquad y, z \in X. \tag{1·1}$$

Then (X, d) is 0-hyperbolic if at least two of $(x \cdot y)_t$, $(y \cdot z)_t$ and $(x \cdot z)_t$ are equal, and not greater than the third, for all $x, y, z, t \in X$. (See [**2**, Section 2, Chapter 1] for more details.) We shall make use of the interpretation of the function $c(g, h)$ in (i) as $d(x, w) = (gx \cdot hx)_x$, where $[x, gx] \cap [x, hx] = [x, w]$.

Finally, we shall be dealing with actions in which every non-identity element acts as a hyperbolic isometry, so we note the following criterion.

LEMMA 1.2. *Suppose a group G acts by isometries on a Λ-tree (X, d) and let $x \in X$ be any point. Then an element $g \in G$ acts as a hyperbolic isometry of X if, and only if, $L_x(g^2) > L_x(g)$.*

Proof. This follows from [**2**, Chapter 3, Lemma 1.8].

Remark. Following [**7**], a length function $L\colon G \to \Lambda$ such that $L(g^2) > L(g)$ for all non-trivial elements $g \in G$ will be called *free*. (Lyndon used the term *archimedean*.)

2. *The rewriting system $\mathcal{R}$*

Let G be a group acting by isometries on a Λ-tree $\mathbf{X} = (X, d)$. We shall assume that the action of G is free and without inversions. Let $B = \{b_i : i \in I\}$ be a set of representatives for the G-orbits. We assume that $0 \in I$, and we shall take b_0 as basepoint in $\mathbf{X}$. It is probably implicit, but we assume that the map $i \mapsto b_i$ is bijective, and that $X \cap I = \varnothing$.

We consider words $\mathbf{x} = s_1 s_2 \ldots s_n$ ($s_i \in X \cup I$) over the alphabet $X \cup I$; we shall usually separate the letters s_i by commas to improve readability, and often add parentheses at the ends of a word. Let $\mathcal{R}$ be the set consisting of the following rewrite rules.

(1) $gb_i, i, x \longrightarrow gx$, for $g \in G$, $i \in I$, and $x \in X$;
(2) $i, b_i \longrightarrow \varepsilon$, where ε is the empty word, for all $i \in I$.

In the usual way, $\mathcal{R}$ induces a relation $\longrightarrow$ on $(X \cup I)^*$, the free monoid generated by the set $X \cup I$, namely the binary relation given by

$$\mathbf{x}_1 \longrightarrow \mathbf{x}_2 \Leftrightarrow \mathbf{x}_2 \text{ results from } \mathbf{x}_1 \text{ by a move of type (1) or type (2).}$$

Let $\overset{*}{\longrightarrow}$ be the reflexive, transitive closure of $\longrightarrow$, and let $\equiv$ be the equivalence relation generated by $\longrightarrow$. Explicitly, we have $\mathbf{x} \equiv \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in (X \cup I)^*$ if, and only if, there exists a sequence

$$\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m = \mathbf{y},$$

where $\mathbf{x}_\mu \in (X \cup I)^*$ for $1 \leqslant \mu \leqslant m$, $m \geqslant 1$, and for $1 \leqslant \mu < m$, either $\mathbf{x}_\mu \longrightarrow \mathbf{x}_{\mu+1}$ or $\mathbf{x}_{\mu+1} \longrightarrow \mathbf{x}_\mu$.

As usual, the *length* of a word

$$\mathbf{x} = (s_1, s_2, \ldots, s_n), \quad s_i \in X \cup I,$$

is n; and a word $\mathbf{x} \in (X \cup I)^*$ is *reduced*, if none of the rewrite rules (1), (2) can be applied to it. We note that, since the rewrite rules shorten the length of a word, $\mathcal{R}$ is *terminating*; that is, for each $\mathbf{x} \in (X \cup I)^*$ there exists $t = t(\mathbf{x}) \in \mathbb{N}_0$ such that no chain of direct moves (1) or (2) applied to $\mathbf{x}$ has more than t terms. Recall that a rewriting system $\mathcal{R}$ on a set S is called *locally confluent* if, whenever $\mathbf{x} \in S$ can be transformed into $\mathbf{y}_1$ by one direct move, and into $\mathbf{y}_2$ by another, then there exists $\mathbf{z} \in S$, such that $\mathbf{z}$ can be reached from each of $\mathbf{y}_1, \mathbf{y}_2$ by an appropriate chain of direct moves.

LEMMA 2.1. *The rewriting system $\mathcal{R}$ on $(X \cup I)^*$ is locally confluent.*

Proof. According to [**5**, Lemma 12.17], it suffices to show that, for each pair of rewrite rules $u_1 \longrightarrow t_1$ and $u_2 \longrightarrow t_2$, the following are satisfied,

(i) if $u_1 = rs$ and $u_2 = st$ with $r, s, t \in (X \cup I)^*$ and $s \neq \varepsilon$, then there exists $w \in (X \cup I)^*$ such that $t_1 t \overset{*}{\longrightarrow} w$ and $rt_2 \overset{*}{\longrightarrow} w$;

(ii) if $u_1 = rst$ and $u_2 = s$ with $r, s, t \in (X \cup I)^*$ and $s \neq \varepsilon$, then there exists $w \in (X \cup I)^*$ such that $t_1 \xrightarrow{*} w$ and $rt_2t \xrightarrow{*} w$.

Note that in our case, $u_1 = u_2$ implies $t_1 = t_2$, so that (i) and (ii) are trivially satisfied (but this makes essential use of the fact that the action of G is free). Thus, we may assume that $u_1 \neq u_2$. To verify (i), if both rewrite rules are of type (1) above, they have the form $gb_i, i, hb_j \longrightarrow ghb_j$ and $hb_j, j, z \longrightarrow hz$ (so $s = hb_j$, etc.). Then it is easy to see that we may take $w = ghz$. It is impossible that both rules are of type (2), so assume one is of type (1) and the other is of type (2). There are two possibilities; the first is that the two rules have the form $gb_i, i, b_i \longrightarrow gb_i$ and $i, b_i \longrightarrow \varepsilon$ (so $s = (i, b_i)$). Then we may take $w = gb_i$. The second is that the rules have the form $i, b_i \longrightarrow \varepsilon$ and $b_i, i, y \longrightarrow y$ (so $s = b_i$, $t_1 = \varepsilon$ and $t_2 = y$, etc.) Then we can put $w = (i, y)$.

To verify (ii), the only possibility is that $u_1 \longrightarrow t_1$ is of type (1) and $u_2 \longrightarrow t_2$ is of type (2), and they have the form $gb_i, i, b_i \longrightarrow gb_i$ and $i, b_i \longrightarrow \varepsilon$ respectively. Thus $s = (i, b_i)$, $r = gb_i$ and $t = \varepsilon$. Then we can let $w = gb_i$.

PROPOSITION 2.2. *Each equivalence class of $(X \cup I)^*$ under $\equiv$ contains exactly one reduced word.*

Proof. Since the rewriting system $\mathcal{R}$ on $(X \cup I)^*$ is terminating, and locally confluent by Lemma 2.1, this follows by a well-known result of M. H. A. Newman (the "Diamond Lemma"); cf. [**10**], or [**4**, Chapter 1, Lemma 5.1].

3. *The group $\hat{G}$*

Consider the subset

$$S := \{(x_0, i_1, x_1, \ldots, i_n, x_n) : n \geqslant 0,\ x_\mu \in X\ (0 \leqslant \mu \leqslant n),\ i_\nu \in I\ (1 \leqslant \nu \leqslant n)\},$$

of $(X \cup I)^*$, and let $\sim$ be the equivalence relation induced by $\mathcal{R}$ on the set S; that is, we have $\mathbf{x} \sim \mathbf{y}$ if, and only if, there exists a finite sequence

$$\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m = \mathbf{y},$$

where $m \geqslant 1$, $\mathbf{x}_\mu \in S$ for $1 \leqslant \mu \leqslant m$, and, for $1 \leqslant \mu < m$, either $\mathbf{x}_\mu \longrightarrow \mathbf{x}_{\mu+1}$ or $\mathbf{x}_{\mu+1} \longrightarrow \mathbf{x}_\mu$. Clearly, the equivalence class $[\mathbf{x}]$ of a word $\mathbf{x} \in S$ under $\sim$ is contained in the equivalence class of $\mathbf{x}$ under the relation $\equiv$; in particular, $[\mathbf{x}]$ contains at most one reduced word by Proposition 2.2. Also, each class $[\mathbf{x}]$ does contain a reduced word, since the rewrite rules (1) and (2) both shorten the length, and S is closed under moves of types (1) and (2).

We define a binary operation on S by

$$\mathbf{x}.\mathbf{y} := \text{ the concatenation } \mathbf{x}, 0, \mathbf{y}.$$

This operation is clearly associative, and $\sim$ is a congruence on S, so $S/\sim$ is a semigroup, with respect to the operation $[\mathbf{x}][\mathbf{y}] = [\mathbf{x}.\mathbf{y}]$. Further, we have, for $\mathbf{x} \in S$,

$$\mathbf{x}.(b_0) = \mathbf{x}, 0, b_0 \xrightarrow{(2)} \mathbf{x} \xleftarrow{(1)} b_0, 0, \mathbf{x} = (b_0).\mathbf{x},$$

so that $[(b_0)]$ is a (two-sided) identity element for $S/\sim$. Moreover, for

$$\mathbf{x} = (g_0 b_{j_0}, i_1, g_1 b_{j_1}, i_2, \ldots, i_n, g_n b_{j_n}) \in S, \tag{3·1}$$

let

$$\bar{\mathbf{x}} := (b_0, j_n, g_n^{-1} b_{i_n}, j_{n-1}, g_{n-1}^{-1} b_{i_{n-1}}, \ldots, j_1, g_1^{-1} b_{i_1}, j_0, g_0^{-1} b_0). \tag{3·2}$$

A straightforward calculation, involving an alternating chain of moves, then shows that

$$[\mathbf{x}][\bar{\mathbf{x}}] = [(b_0)] = [\bar{\mathbf{x}}][\mathbf{x}];$$

hence, $S/\sim$ is a group, which we denote by $\hat{G}$. It is easy to see that $\hat{G}$ is independent, up to isomorphism, of the system of representatives for the G-orbits used in its definition.

4. *The length function L*

For a reduced word $\mathbf{x} \in S$ as in (3·1), set

$$L([\mathbf{x}]) := d(b_0, g_0 b_{j_0}) + \sum_{k=1}^{n} d(b_{i_k}, g_k b_{j_k}).$$

Since each equivalence class of S contains a unique reduced word, this defines a map $L\colon \hat{G} \longrightarrow \Lambda$. We shall show that L is a Lyndon length function on $\hat{G}$.

LEMMA 4.1. *For* $[\mathbf{x}] \in \hat{G}$, *we have* $L([\mathbf{x}]) = L([\mathbf{x}]^{-1})$.

Proof. Let $\mathbf{x} \in S$ be reduced and as in (3·1), and consider $\bar{\mathbf{x}}$. If, for some k with $1 \leqslant k \leqslant n$, a type (1) move

$$g_k^{-1} b_{i_k}, j_{k-1}, g_{k-1}^{-1} b_{i_{k-1}} \longrightarrow g_k^{-1} g_{k-1}^{-1} b_{i_{k-1}}$$

were possible in $\bar{\mathbf{x}}$, then it would follow that $j_{k-1} = i_k$, so that the type (1) move

$$g_{k-1} b_{j_{k-1}}, i_k, g_k b_{j_k} \longrightarrow g_{k-1} g_k b_{j_k}$$

would be possible in $\mathbf{x}$, contradicting the fact that $\mathbf{x}$ is reduced. Hence, the only possible type (1) move in $\bar{\mathbf{x}}$ is

$$b_0, j_n, g_n^{-1} b_{i_n} \longrightarrow g_n^{-1} b_{i_n}, \quad j_n = 0. \tag{4·1}$$

Similarly, we see that the only possible type (2) move in $\bar{\mathbf{x}}$ is

$$j_0, g_0^{-1} b_0 \longrightarrow \varepsilon, \quad j_0 = 0 \text{ and } g_0 = 1. \tag{4·2}$$

Moreover, carrying out these moves, where possible, gives the reduced word in the equivalence class $[\mathbf{x}]^{-1}$. Using the facts that $d(b_0, b_0) = 0$, that G acts by isometries, and that d is symmetric, the result follows now by a straightforward computation.

Next, we shall calculate $c([\mathbf{x}], [\mathbf{y}])$ for $[\mathbf{x}], [\mathbf{y}] \in \hat{G}$. Our result is as follows.

LEMMA 4.2. *Let*

$$\mathbf{x} = (x_0, i_1, x_1, \ldots, i_m, x_m), \ \mathbf{y} = (y_0, j_1, y_1, \ldots, j_n, y_n) \in S$$

be reduced words. Let $k \geqslant 0$ *be maximal with respect to the condition that*

$$(x_0, i_1, x_1, \ldots, i_k) = (y_0, j_1, y_1, \ldots, j_k),$$

and let

$$\mathbf{w} = \mathbf{w}(\mathbf{x}, \mathbf{y}) := \begin{cases} (x_0, i_1, x_1, \ldots, x_{k-1}), & k \geqslant 1; \\ \varepsilon, & k = 0. \end{cases}$$

Then

$$c([\mathbf{x}], [\mathbf{y}]) = L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_k \cdot y_k)_{b_{i_k}} \tag{4·3}$$

where $L([\mathbf{w}(\mathbf{x}, \mathbf{y})])$ *has to be interpreted as* 0 *if* $\mathbf{w} = \varepsilon$, $i_0 := 0$, *and, for points* $x, y, b \in X$, $(x \cdot y)_b$ *is as in* (1·1); *in particular, we have* $c([\mathbf{x}], [\mathbf{y}]) \geqslant L([\mathbf{w}(\mathbf{x}, \mathbf{y})])$ *and* $c([\mathbf{x}], [\mathbf{y}]) \in \Lambda$.

Proof. The fact that $c([\mathbf{x}], [\mathbf{y}]) \in \Lambda$ follows from (4·3) together with [**2**, Chapter 2, Lemma 1.6]. Since $c([\mathbf{x}], [\mathbf{y}])$ is symmetric by Lemma 4.1, we may suppose without loss of generality that $m \leqslant n$. Let

$$\mathbf{x}' := \begin{cases} (b_0, i_k, x_k, \ldots, i_m, x_m), & k \geqslant 1 \\ \mathbf{x}, & k = 0 \end{cases} \quad \text{and} \quad \mathbf{y}' := \begin{cases} (b_0, j_k, y_k, \ldots, j_n, y_n), & k \geqslant 1 \\ \mathbf{y}, & k = 0. \end{cases}$$

We note that $\mathbf{x}', \mathbf{y}'$ are reduced words, unless $k \geqslant 1$ and either $i_k = 0$ or $j_k = 0$, in which case we can use the type (1) moves

$$b_0, 0, x_k \longrightarrow x_k \quad \text{and} \quad b_0, 0, y_k \longrightarrow y_k$$

to delete b_0, i_k or b_0, j_k where possible, after which the resulting words are reduced. It follows from this discussion and a computation similar to one occurring in the proof of Lemma 4.1 that

$$L([\mathbf{x}]) = L([\mathbf{w}]) + L([\mathbf{x}'])$$

and

$$L([\mathbf{y}]) = L([\mathbf{w}]) + L([\mathbf{y}']).$$

Also, we have $\mathbf{x} \sim \mathbf{w}.\mathbf{x}'$ and $\mathbf{y} \sim \mathbf{w}.\mathbf{y}'$, hence $\bar{\mathbf{x}}.\mathbf{y} \sim \overline{\mathbf{x}'}.\mathbf{y}'$; here, the concatenations $\varepsilon.\mathbf{x}$ and $\varepsilon.\mathbf{y}$ are to be interpreted as $\mathbf{x}$ and $\mathbf{y}$, respectively. We now concentrate on the case when $k < m$. Writing $x_\ell = g_\ell b_{\mu_\ell}$ for $0 \leqslant \ell \leqslant m$, we claim that

$$\overline{\mathbf{x}'}.\mathbf{y}' \sim (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, g_{k+1}^{-1} b_{i_{k+1}}, \mu_k, g_k^{-1} y_k, j_{k+1}, y_{k+1}, \ldots, j_n, y_n). \tag{4·4}$$

Indeed, if $k > 0$, then

$$\begin{aligned} \overline{\mathbf{x}'}.\mathbf{y}' &= \overline{(1 \cdot b_0, i_k, g_k b_{\mu_k}, \ldots, i_m, g_m b_{\mu_m})}.(b_0, j_k, y_k, \ldots, j_n, y_n) \\ &= (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, \mu_k, g_k^{-1} b_{i_k}, 0, b_0).(b_0, j_k, y_k, \ldots, j_n, y_n) \\ &\sim (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, \mu_k, g_k^{-1} b_{i_k}, j_k, y_k, \ldots, j_n, y_n) \\ &\sim (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, g_{k+1}^{-1} b_{i_{k+1}}, \mu_k, g_k^{-1} y_k, j_{k+1}, y_{k+1}, \ldots, j_n, y_n), \end{aligned}$$

where we have used the fact that $i_k = j_k$ for $k > 0$ in the last step. If, on the other hand, $k = 0$, then

$$\begin{aligned} \overline{\mathbf{x}'}.\mathbf{y}' = \bar{\mathbf{x}}.\mathbf{y} &= (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, \mu_1, g_1^{-1} b_{i_1}, \mu_0, g_0^{-1} b_0).(y_0, j_1, y_1, \ldots, j_n, y_n) \\ &\sim (b_0, \mu_m, g_m^{-1} b_{i_m}, \mu_{m-1}, \ldots, \mu_1, g_1^{-1} b_{i_1}, \mu_0, g_0^{-1} y_0, j_1, y_1, \ldots, j_n, y_n), \end{aligned}$$

which coincides with the right-hand side of (4·4) in the case when $k = 0$, as desired.

The only possible type (1) move in the word on the right-hand side of (4·4) is

$$b_0, \mu_m, g_m^{-1} b_{i_m} \longrightarrow g_m^{-1} b_{i_m}, \quad \mu_m = 0,$$

while the only possible type (2) move is

$$\mu_k, g_k^{-1} y_k \longrightarrow \varepsilon, \quad x_k = y_k.$$

This follows from a discussion of possible type (1) and type (2) moves similar to one in the proof of Lemma 4.1, using the fact that $\mathbf{x}$ and $\mathbf{y}$ are reduced. Moreover, having performed these moves in (4·4) where possible, the resulting word is reduced, since $k < m$ and therefore $i_{k+1} \neq j_{k+1}$ by choice of k. It follows that

$$\begin{aligned} L([\mathbf{x}]^{-1}[\mathbf{y}]) &= L([\overline{\mathbf{x}'}.\mathbf{y}']) \\ &= \sum_{\ell=k+1}^{m} d(b_{i_\ell}, x_\ell) + d(x_k, y_k) + \sum_{\ell=k+1}^{n} d(b_{j_\ell}, y_\ell) \\ &= L([\mathbf{x}']) - d(b_{i_k}, x_k) + d(x_k, y_k) + L([\mathbf{y}']) - d(b_{j_k}, y_k), \end{aligned}$$

and therefore

$$\begin{aligned} c([\mathbf{x}], [\mathbf{y}]) &= L([\mathbf{w}]) + \tfrac{1}{2}\{d(b_{i_k}, x_k) + d(b_{j_k}, y_k) - d(x_k, y_k)\} \\ &= L([\mathbf{w}]) + (x_k \cdot y_k)_{b_{i_k}}, \end{aligned}$$

as claimed.

It remains to deal with the case when $k = m$. Again writing $x_m = g_m b_{\mu_m}$, we have, for $m \geqslant 0$,

$$\overline{\mathbf{x}'}.\mathbf{y}' \sim (b_0, \mu_m, g_m^{-1} y_m, j_{m+1}, y_{m+1}, \ldots, j_n, y_n).$$

The only possible type (1) move now is

$$b_0, \mu_m, g_m^{-1} y_m \longrightarrow g_m^{-1} y_m, \quad \mu_m = 0, \tag{4·5}$$

while the only possible type (2) move is

$$\mu_m, g_m^{-1} y_m \longrightarrow \varepsilon, \quad x_m = y_m. \tag{4·6}$$

If $\mu_m = 0$, performing the type (1) move (4·5) results in a reduced word, whereas for $\mu_m \neq 0$ and $x_m = y_m$, performing the type (2) move (4·6) results in a reduced word for $j_{m+1} \neq 0$, while for $j_{m+1} = 0$ a further type (1) move

$$b_0, j_{m+1}, y_{m+1} \longrightarrow y_{m+1}$$

is necessary to reach a reduced form. Given this, a routine calculation yields that, for $k = m \geqslant 0$,

$$L([\mathbf{x}]^{-1}[\mathbf{y}]) = L([\overline{\mathbf{x}'}.\mathbf{y}']) = d(x_m, y_m) + \sum_{\ell=m+1}^{n} d(b_{j_\ell}, y_\ell),$$

and the rest of the proof proceeds as before.

We are now ready for the main result of this section.

PROPOSITION 4.3. *The mapping L is a Lyndon length function on $\hat{G}$ with $c(g, h) \in \Lambda$ for all $g, h \in \hat{G}$.*

Proof. In view of Lemmas 4.1 and 4.2, we only need to show that, if

$$\mathbf{x} = (x_0, i_1, x_1, \ldots, i_\ell, x_\ell),\ \mathbf{y} = (y_0, i'_1, y_1, \ldots, i'_m, y_m),\ \mathbf{z} = (z_0, i''_1, z_1, \ldots, i''_n, z_n) \in S$$

are reduced words, then at least two of the three values

$$c([\mathbf{x}], [\mathbf{y}]),\ c([\mathbf{y}], [\mathbf{z}]),\ c([\mathbf{x}], [\mathbf{z}])$$

are equal, and less than or equal to the third. Let $\mathbf{p}(\mathbf{x}, \mathbf{y})$ denote the largest common prefix of the words $\mathbf{x}$ and $\mathbf{y}$, with a similar definition applying to the pairs $\mathbf{y}, \mathbf{z}$ and $\mathbf{x}, \mathbf{z}$. Without loss of generality we can assume that $\mathbf{p}(\mathbf{x}, \mathbf{y}) = \mathbf{p}(\mathbf{y}, \mathbf{z})$ is a prefix of $\mathbf{p}(\mathbf{x}, \mathbf{z})$. We distinguish two main cases according to whether $\mathbf{p}(\mathbf{x}, \mathbf{y})$ ends in a point or an index.

Case (a). *We have* $\mathbf{p}(\mathbf{x}, \mathbf{y}) = x_0, i_1, x_1, \ldots, x_{\kappa-1}$ *for some* $\kappa \geqslant 1$ (i.e., $\mathbf{p}(\mathbf{x}, \mathbf{y})$ is non-empty and ends in a point). Then, in the notation of Lemma 4.2, $k = \kappa - 1$, and so

$$\mathbf{w}(\mathbf{x}, \mathbf{y}) = \begin{cases} (x_0, i_1, x_1, \ldots, x_{\kappa-2}), & \kappa \geqslant 2 \\ \varepsilon, & \kappa = 1, \end{cases}$$

and, by (4·3),

$$\begin{aligned} c([\mathbf{x}], [\mathbf{y}]) &= L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_{\kappa-1} \cdot y_{\kappa-1})_{b_{i_{\kappa-1}}} \\ &= L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + d(b_{i_{\kappa-1}}, x_{\kappa-1}) \\ &= L([\mathbf{p}(\mathbf{x}, \mathbf{y})]) \\ &= c([\mathbf{y}], [\mathbf{z}]). \end{aligned}$$

Moreover, if $i_\kappa \neq i''_\kappa$, we have $\mathbf{p}(\mathbf{x}, \mathbf{y}) = \mathbf{p}(\mathbf{x}, \mathbf{z})$, and hence $\mathbf{w}(\mathbf{x}, \mathbf{y}) = \mathbf{w}(\mathbf{x}, \mathbf{z})$, whereas for $i_\kappa = i''_\kappa$, the string $\mathbf{p}(\mathbf{x}, \mathbf{y})$ is a prefix of $\mathbf{w}(\mathbf{x}, \mathbf{z})$. Consequently, for $i_\kappa \neq i''_\kappa$,

$$\begin{aligned} c([\mathbf{x}], [\mathbf{z}]) &= L([\mathbf{w}(\mathbf{x}, \mathbf{z})]) + (x_{\kappa-1} \cdot z_{\kappa-1})_{b_{i_{\kappa-1}}} \\ &= L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_{\kappa-1} \cdot y_{\kappa-1})_{b_{i_{\kappa-1}}} \\ &= c([\mathbf{x}], [\mathbf{y}]) \\ &= c([\mathbf{y}], [\mathbf{z}]), \end{aligned}$$

while, for $i_\kappa = i''_\kappa$,

$$c([\mathbf{x}], [\mathbf{z}]) \geqslant L([\mathbf{w}(\mathbf{x}, \mathbf{z})]) \geqslant L([\mathbf{p}(\mathbf{x}, \mathbf{y})]) = c([\mathbf{x}], [\mathbf{y}]) = c([\mathbf{y}], [\mathbf{z}]).$$

Case (b). *We have* $\mathbf{p}(\mathbf{x}, \mathbf{y}) = x_0, i_1, \ldots, x_{\kappa-1}, i_\kappa$ *for some* $\kappa \geqslant 0$ (i.e., $\mathbf{p}(\mathbf{x}, \mathbf{y})$ is empty or ends in an index). In this case, $\mathbf{w}(\mathbf{x}, \mathbf{y}) = \mathbf{w}(\mathbf{y}, \mathbf{z})$ is a prefix of $\mathbf{w}(\mathbf{x}, \mathbf{z})$, and we have

$$c([\mathbf{x}], [\mathbf{y}]) = L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_\kappa \cdot y_\kappa)_{b_{i_\kappa}}$$

as well as

$$c([\mathbf{y}], [\mathbf{z}]) = L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (y_\kappa \cdot z_\kappa)_{b_{i_\kappa}}.$$

We now distinguish two subcases.

Case (b_1). Suppose that $x_\kappa = z_\kappa$. Then

$$c([\mathbf{x}], [\mathbf{y}]) = c([\mathbf{y}], [\mathbf{z}])$$

and

$$\begin{aligned} c([\mathbf{x}], [\mathbf{z}]) &\geqslant L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + d(b_{i_\kappa}, x_\kappa) \\ &\geqslant L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_\kappa \cdot y_\kappa)_{b_{i_\kappa}} = c([\mathbf{x}], [\mathbf{y}]), \end{aligned}$$

since, by the triangle inequality,

$$\begin{aligned}(x_\kappa \cdot y_\kappa)_{b_{i_\kappa}} &= \tfrac{1}{2}\{d(b_{i_\kappa}, x_\kappa) + d(b_{i_\kappa}, y_\kappa) - d(x_\kappa, y_\kappa)\}\\ &\leqslant \tfrac{1}{2}\{d(b_{i_\kappa}, x_\kappa) + d(b_{i_\kappa}, y_\kappa) - d(b_{i_\kappa}, y_\kappa) + d(b_{i_\kappa}, x_\kappa)\} = d(b_{i_\kappa}, x_\kappa).\end{aligned}$$

Case (b_2). Suppose that $x_\kappa \neq z_\kappa$. Then $\mathbf{p}(\mathbf{x}, \mathbf{z}) = \mathbf{p}(\mathbf{x}, \mathbf{y})$, so $\mathbf{w}(\mathbf{x}, \mathbf{z}) = \mathbf{w}(\mathbf{x}, \mathbf{y})$, and

$$c([\mathbf{x}], [\mathbf{z}]) = L([\mathbf{w}(\mathbf{x}, \mathbf{y})]) + (x_\kappa \cdot z_\kappa)_{b_{i_\kappa}}.$$

In this case, the desired result follows since $\mathbf{X}$ is 0-hyperbolic, so two of the quantities

$$(x_\kappa \cdot y_\kappa)_{b_{i_\kappa}},\ (y_\kappa \cdot z_\kappa)_{b_{i_\kappa}},\ (x_\kappa \cdot z_\kappa)_{b_{i_\kappa}}$$

are equal, and less than or equal to the third. This completes the proof.

Given the last result, it follows from of Theorem 1.1 (ii) that there exists a Λ-tree $\widehat{\mathbf{X}} = (\widehat{X}, \hat{d})$ on which $\hat{G}$ acts by isometries, with a basepoint b such that $L = L_b$ is the displacement length with respect to b.

5. *The action of $\hat{G}$ on $\widehat{\mathbf{X}}$*

LEMMA 5.1. *The action of $\hat{G}$ on $\widehat{\mathbf{X}}$ is free and without inversions.*

Proof. We need to show that if $g \in \hat{G}$, $g \neq 1$, then g acts as a hyperbolic isometry on $\widehat{\mathbf{X}}$; that is, $L(g^2) > L(g)$. Let $g = [\mathbf{x}]$, where $\mathbf{x} \in S$ is reduced, and write

$$\mathbf{x} = (g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}),$$

as previously. We use induction on n. We have

$$\mathbf{x}.\mathbf{x} = (g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}, 0, g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}), \tag{5·1}$$

and this word is reduced, unless either (1) $g_0 = 1$ and $j_0 = 0$, or (2) $j_n = 0$. If the right-hand side of (5·1) is reduced, then

$$L(g^2) \geqslant L(g) + d(b_0, g_0 b_{j_0}) > L(g),$$

as (1) does not hold, so g is hyperbolic. Otherwise we have two cases to consider.

Case (a). Suppose that $g_0 = 1$ and $j_0 = 0$. Then $n \geqslant 1$ (since $g \neq 1$), and

$$\mathbf{x}.\mathbf{x} \sim (b_0, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}). \tag{5·2}$$

If $j_n \neq i_1$ then the word on the right-hand side of (5·2) is reduced, and, since $\mathbf{x}$ is reduced, we have

$$L(g^2) \geqslant L(g) + d(b_{i_1}, g_1 b_{j_1}) > L(g),$$

so again g is hyperbolic. Next, suppose that $j_n = i_1$. Then, if $n = 1$, we have

$$\mathbf{x}.\mathbf{x} \sim (b_0, i_1, g_1 b_{i_1}).(b_0, i_1, g_1 b_{i_1}) \sim (b_0, i_1, g_1^2 b_{i_1}),$$

and

$$L(g^2) = d(b_{i_1}, g_1^2 b_{i_1}) > d(b_{i_1}, g_1 b_{i_1}) = L(g)$$

in this case, because $g_1 \neq 1$ as $\mathbf{x}$ is reduced, and G acts freely and without inversions. Hence g is hyperbolic. Now assume that $n \geqslant 2$; then

$$\mathbf{x}.\mathbf{x} \sim (b_0, i_1, g_1 b_{j_1}, \ldots, i_n, g_n g_1 b_{j_1}, i_2, \ldots, i_n, g_n b_{j_n}).$$

If either $i_n \neq j_1$ or $g_n \neq g_1^{-1}$, then this is reduced, and hence

$$L(g^2) \geqslant L(g) + d(b_{i_n}, g_n g_1 b_{j_1}) > L(g),$$

so g is hyperbolic.

Suppose $i_n = j_1$ and $g_n = g_1^{-1}$. Then $n \geqslant 3$ as $\mathbf{x}$ is reduced; in this situation, let

$$\mathbf{u} := (b_0, i_1, g_1 b_{j_1}) \quad \text{and} \quad \mathbf{x}' := (b_0, i_2, g_2 b_{j_2}, \ldots, g_{n-1} b_{j_{n-1}}).$$

Then

$$\overline{\mathbf{u}} = (b_0, j_1, g_1^{-1} b_{i_1}, 0, b_0) \sim (b_0, j_1, g_1^{-1} b_{i_1}),$$

and we find that $\mathbf{x} \sim \mathbf{u}.\mathbf{x}'.\overline{\mathbf{u}}$. It follows that $g = [\mathbf{u}][\mathbf{x}'][\mathbf{u}]^{-1}$, so $[\mathbf{x}'] \neq 1$ since $g \neq 1$; and, by induction, we infer that $[\mathbf{x}']$ is hyperbolic, hence so is the element g.

Case (b). Suppose that $j_n = 0$. If $n = 0$, then

$$\mathbf{x}.\mathbf{x} \sim (g_0 b_0, 0, g_0 b_0) \sim (g_0^2 b_0),$$

and $g_0 \neq 1$ as $g \neq 1$, so

$$L(g^2) = d(b_0, g_0^2 b_0) > d(b_0, g_0 b_0) = L(g),$$

since G acts freely and without inversions. Assume that $n \geqslant 1$. Then

$$\mathbf{x}.\mathbf{x} \sim (g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_n, g_n b_{j_n}).$$

This is reduced unless $i_n = j_0$ and $g_n = g_0^{-1}$, and if it is reduced, then

$$L(g^2) \geqslant L(g) + d(b_{i_n}, g_n g_0 b_{j_0}) > L(g);$$

thus, g is hyperbolic.

Suppose that $i_n = j_0$ and $g_n = g_0^{-1}$. Then $n \geqslant 2$, otherwise $\mathbf{x} \sim (b_0)$, contradicting our hypothesis that $g \neq 1$. We have

$$\mathbf{x}.\mathbf{x} \sim (g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_{n-1}, g_{n-1} b_{j_{n-1}}, i_1, g_1 b_{j_1}, \ldots, j_0, g_n b_{j_n}).$$

If $i_1 \neq j_{n-1}$ then this is reduced, and, since $\mathbf{x}$ is reduced, we obtain

$$L(g^2) \geqslant L(g) + d(b_{i_1}, g_1 b_{j_1}) > L(g);$$

hence, g is again hyperbolic. It remains to consider the case when $j_n = 0$, $i_n = j_0$, $g_n = g_0^{-1}$, and $i_1 = j_{n-1}$. Let $\mathbf{u} := (g_0 b_{j_0}, i_1, b_0)$, and set

$$\mathbf{x}' := \begin{cases} (g_1 b_0), & n = 2 \\ (g_1 b_{j_1}, i_2, g_2 b_{j_2}, \ldots, i_{n-1}, g_{n-1} b_0), & n \geqslant 3. \end{cases}$$

Then

$$\overline{\mathbf{u}} = (b_0, 0, b_{i_1}, j_0, g_0^{-1} b_0) \sim (b_{i_1}, j_0, g_0^{-1} b_0),$$

and it is easily checked that $\mathbf{x} \sim \mathbf{u}.\mathbf{x}'.\overline{\mathbf{u}}$. It follows again that $g = [\mathbf{u}][\mathbf{x}'][\mathbf{u}]^{-1}$, so $[\mathbf{x}'] \neq 1$, and by induction $[\mathbf{x}']$ is hyperbolic, hence so is g. This complete the proof of the lemma.

We call a Lyndon length function $L\colon G \to \Lambda$ *strongly regular* if, for each $g \in G$ and every $\gamma \in \Lambda$ such that $0 \leqslant \gamma \leqslant L(g)$, there exist elements $g_1, g_2 \in G$ such that $g = g_1 g_2$, $L(g_1) = \gamma$, and $L(g_1) + L(g_2) = L(g)$. Our next result spells out the precise connection between strong regularity of a length function on a group G and the associated action of G on the corresponding Λ-tree.

LEMMA 5.2. *Suppose that a group G acts by isometries on a Λ-tree $\mathbf{X} = (X, d)$, and let $x_0 \in X$ be any point. Then the following assertions are equivalent*:

(i) *the group G is transitive on the subtree spanned by the orbit of x_0*;

(ii) *the displacement function L_{x_0} is strongly regular.*

Proof. See [**3**, Section A.3, Proposition A.32].

Remark. The reader may wonder why we have introduced the term *strongly regular* here. The reason is that there exists already a notion of *regular* length function in the literature, which has proved useful; cf., for instance, [**7**]: a Lyndon length function $L\colon G \to \Lambda$ on a group G is called *regular* if, for any two elements $g, h \in G$, there exist elements $u, g_1, h_1 \in G$ such that $g = ug_1$, $h = uh_1$, $L(g) = L(u) + L(g_1)$, $L(h) = L(u) + L(h_1)$, and $L(u) = c(g, h)$. It is shown in [**3**, Section A.3] that a strongly regular length function $L\colon G \to \Lambda$ with the property that $c(g, h) \in \Lambda$ for all $g, h \in G$ is in fact regular; thus, in particular, justifying our terminology (cf. [**3**, Proposition A.30]).

LEMMA 5.3. *The action of $\hat{G}$ on $\widehat{\mathbf{X}}$ is transitive.*

Proof. It follows from [**2**, Chapter 2, Theorem 4.6] (with $X' = Z = \widehat{X}$ and $w = b$, so μ is the identity map), that $\widehat{\mathbf{X}}$ is spanned by the orbit $\hat{G}b$. In view of the previous lemma, it therefore suffices to show that $L = L_b$ is strongly regular.

Let $\mathbf{x}$ be a reduced word as in (3·1), and let $\gamma \in \Lambda$ be such that $0 \leqslant \gamma \leqslant L([\mathbf{x}])$. Set $i_0 := 0$, so that $L([\mathbf{x}]) = \sum_{k=0}^{n} d(b_{i_k}, g_k b_{j_k})$. Let $N \in \mathbb{N}_0$ be minimal subject to the condition that $\gamma \leqslant \sum_{k=0}^{N} d(b_{i_k}, g_k b_{j_k})$, and let $\delta := \gamma - \sum_{k=0}^{N-1} d(b_{i_k}, g_k b_{j_k})$, thus $0 \leqslant \delta \leqslant d(b_{i_N}, g_N b_{j_N})$, and $\delta > 0$ for $N \geqslant 1$.

Let z be the point on the segment $[b_{i_N}, g_N b_{j_N}]$ of $\mathbf{X}$ at distance δ from b_{i_N}, and define

$$\mathbf{y} = \begin{cases} (g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_{N-1}, g_{N-1} b_{j_{N-1}}, i_N, z), & N \geqslant 1 \\ (z), & N = 0. \end{cases}$$

Then $\mathbf{y} \in S$ is reduced, and

$$L([\mathbf{y}]) = \sum_{k=0}^{N-1} d(b_{i_k}, g_k b_{j_k}) + d(b_{i_N}, z) = \sum_{k=0}^{N-1} d(b_{i_k}, g_k b_{j_k}) + \delta = \gamma.$$

By Lemma 4.2,

$$\begin{aligned} c([\mathbf{x}], [\mathbf{y}]) &= L([(g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_{N-1}, g_{N-1} b_{j_{N-1}})]) + (g_N b_{j_N} \cdot z)_{b_{i_N}} \\ &= L([(g_0 b_{j_0}, i_1, g_1 b_{j_1}, \ldots, i_{N-1}, g_{N-1} b_{j_{N-1}})]) + d(b_{i_N}, z) \\ &= L([\mathbf{y}]). \end{aligned}$$

Letting $g := [\mathbf{x}]$, $g_1 := [\mathbf{y}]$, and $g_2 := g_1^{-1} g$, we have found that $L(g_1) = \gamma$ and

$$L(g_2) = L([\mathbf{y}]^{-1}[\mathbf{x}]) = L([\mathbf{x}]^{-1}[\mathbf{y}]) = L([\mathbf{x}]) - L([\mathbf{y}]) = L(g) - L(g_1),$$

which shows that L is strongly regular, as desired.

We are now in a position to establish our first main result.

THEOREM 5.4. *Let G be a group acting freely and without inversions on a Λ-tree $\mathbf{X}$. Then there exists a group $\hat{G}$ acting freely, without inversions, and transitively on a Λ-tree $\widehat{\mathbf{X}}$, together with a group embedding $\varphi\colon G \longrightarrow \hat{G}$ and a G-equivariant isometry $\mu\colon \mathbf{X} \longrightarrow \widehat{\mathbf{X}}$.*

Proof. Define $\varphi\colon G \longrightarrow \hat{G}$ by $\varphi(g) = [(gb_0)]$. Then φ is a group homomorphism since, for $g, h \in G$,

$$\varphi(g)\varphi(h) = [(gb_0)][(hb_0)] = [(gb_0).(hb_0)] = [(gb_0, 0, hb_0)] = [(ghb_0)] = \varphi(gh),$$

where we have used the type (1) move $gb_0, 0, hb_0 \longrightarrow ghb_0$ in the next to last step. Moreover, since the action of G on $\mathbf{X}$ is free, we have, for $g \in G$, that

$$\varphi(g) = 1 \iff [(gb_0)] = [(b_0)] \iff gb_0 = b_0 \iff g = 1,$$

using the fact that (gb_0) and (b_0) are reduced words. Hence, φ is an embedding.

Since the action of $\hat{G}$ on $\widehat{\mathbf{X}}$ is transitive by Lemma 5.3, we have

$$\widehat{X} = \big\{[\mathbf{x}]b : [\mathbf{x}] \in \hat{G}\big\},$$

and the metric $\hat{d}$ on $\widehat{\mathbf{X}}$ is given by

$$\hat{d}([\mathbf{x}]b, [\mathbf{y}]b) = \hat{d}(b, [\mathbf{x}]^{-1}[\mathbf{y}]b) = L([\mathbf{x}]^{-1}[\mathbf{y}]),$$

as $L = L_b$. Define $\mu\colon X \longrightarrow \widehat{X}$ by $\mu(gb_i) := [(gb_i)]b$. Then

$$\begin{aligned}
\hat{d}(\mu(gb_i), \mu(hb_j)) &= L([(gb_i)]^{-1}[(hb_j)])\\
&= L([(b_0, i, g^{-1}b_0)][(hb_j)])\\
&= L([(b_0, i, g^{-1}b_0, 0, hb_j)])\\
&= L([(b_0, i, g^{-1}hb_j)])\\
&= d(b_i, g^{-1}hb_j) \quad \text{(whether or not } (b_0, i, g^{-1}hb_j) \text{ is reduced)}\\
&= d(gb_i, hb_j),
\end{aligned}$$

so μ is an isometry. Finally, μ is G-equivariant as

$$\mu(hgb_i) = [(hgb_i)]b = [(hb_0)][(gb_i)]b = [(hb_0)]\mu(gb_i) = \varphi(h)\mu(gb_i).$$

This completes the proof of the theorem.

A question arising from the results of [**7**] is the following. Can a group with a Lyndon length function L always be embedded in a length-preserving way into a group with a regular Lyndon length function? With an obvious necessary restriction, Theorem 5.4 provides an affirmative answer to this question in the case when L is free.

COROLLARY 5.5. *Let G be a group endowed with a free length function $L\colon G \to \Lambda$ satisfying $c(g, h) \in \Lambda$ for all $g, h \in G$. Then G can be embedded in a length-preserving way into a group with a free, regular length function.*

Proof. By Part (i) of Theorem 1.1 and Lemma 1.2, our hypotheses guarantee existence of a Λ-tree $\mathbf{X} = (X, d)$ on which G acts freely and without inversions, and such that $L = L_{x_0}$ for some point $x_0 \in X$. By Theorem 5.4, G can be embedded into a group $\hat{G}$ acting freely, without inversions, and transitively on a Λ-tree $\widehat{\mathbf{X}}$. Furthermore, $\mathbf{X}$ embeds by means of a

G-equivariant isometry μ into $\widehat{\mathbf{X}}$; in particular, the group embedding is length-preserving with respect to the length function $\hat{L} = L_{\mu(x_0)}$ on $\hat{G}$, the latter being strongly regular by Lemma 5.2, and satisfying

$$c(\hat{g}, \hat{h}) = (\hat{g}\mu(x_0) \cdot \hat{h}\mu(x_0))_{\mu(x_0)} \in \Lambda$$

for all $\hat{g}, \hat{h} \in \hat{G}$ by Lemma 1.6 in [**2**, Chapter 2]. According to the remark preceding Lemma 5.3, $\hat{L}$ is regular, and it is free by Lemma 1.2, whence the result.

6. *Some remarks concerning the* Λ*-tree* $\widehat{\mathbf{X}}$

We shall prove one general result about the structure of $\widehat{\mathbf{X}}$, by calculating the degree of its points. Recall that, if $\mathbf{X} = (X, d)$ is a Λ-tree and $v \in X$, then the set of directions at v is the quotient of the set of segments $\{[v, x] \mid x \in X,\ x \neq v\}$ by the equivalence relation $\equiv$ defined by

$$[v, x] \equiv [v, y] \iff [v, x] \cap [v, y] \neq \{v\} \iff (x \cdot y)_v > 0$$

(cf [**2**, Chapter 2], after Lemma 1.7). The *degree* of v, denoted by $\deg_{\mathbf{X}}(v)$, is the cardinality of the set of directions at v. (Degree is also called valency or index of ramification in the literature.) We denote the equivalence class containing $[v, x]$ by $\langle v, x\rangle$.

PROPOSITION 6.1. *In Theorem* 5.4*, the degree of every point of* $\widehat{\mathbf{X}}$ *is* $\sum_{i\in I} \deg_{\mathbf{X}}(b_i)$.

Proof. Since $\hat{G}$ acts transitively, it is enough to show that $\deg_{\widehat{\mathbf{X}}}(b) = \sum_{i\in I} \deg_{\mathbf{X}}(b_i)$. To do this it suffices to define a bijective map

$$\varphi : \coprod_{i\in I} D_i \longrightarrow D,$$

where D_i is the set of directions at b_i in $\mathbf{X}$, and D is the set of directions at b in $\widehat{\mathbf{X}}$. We set

$$\varphi(\langle b_i, x\rangle) = \langle b, gb\rangle,$$

where $g = [(b_0, i, x)] \in \hat{G}$, for $x \in X, x \neq b_i$. (Note that (b_0, i, x) is reduced unless $i = 0$, when the reduced form of g is $[(x)]$.)

We need to show φ is well-defined and one-to-one. If $g, h \in \hat{G}$, then, as noted in the introduction,

$$(gb \cdot hb)_b = c(g, h).$$

Suppose $g = [(b_0, i, x)]$ and $h = [(b_0, j, y)]$. If $i \neq j$, it follows from Lemma 4.2 and the observation that $(z \cdot b_0)_{b_0} = 0$ for any $z \in X$ that $c(g, h) = 0$. (There are several cases, depending on whether i, j are equal to zero or not.) If $i = j$, then Lemma 4.2 gives $c(g, h) = (x \cdot y)_{b_i}$ (again there are several cases). This shows that φ is indeed well-defined and one-to-one, and it remains to show that it is onto.

Since $\hat{G}$ acts transitively, a direction in D has the form $\langle b, gb\rangle$, where $g \in \hat{G}$, $g \neq 1$, say $g = [(x_0, i_1, x_1, \ldots, i_n, x_n)]$ in reduced form. Suppose $x_0 = hb_j$, where $h \in G$.

If $x_0 \neq b_0$, then $\langle b, gb\rangle = \varphi(\langle b_0, x_0\rangle)$ since $\varphi(\langle b_0, x_0\rangle) = \langle b, [x_0]b\rangle$ and, by Lemma 4.2,

$$c([x_0], g) = (x_0 \cdot x_0)_{b_0} = d(b_0, x_0) > 0.$$

If $x_0 = b_0$, then $n \geqslant 1$ since $g \neq 1$, and $x_1 \neq b_{i_1}$ since the expression for g is reduced. Hence $\langle b, gb \rangle = \varphi(\langle b_{i_1}, x_1 \rangle)$, because $\varphi(\langle b_{i_1}, x_1 \rangle) = \langle b, [b_0, i_1, x_1]b \rangle$ and, by Lemma 4.2,

$$c([(b_0, i_1, x_1)], g) = L([b_0]) + (x_1 \cdot x_1)_{b_{i_1}} = d(b_{i_1}, x_1) > 0,$$

because $(b_0, i_1, x_1) = (x_0, i_1, x_1)$ is reduced. This completes the proof.

If we take $\mathbf{X}$ to be an arbitrary Λ-tree and G to be the trivial group in Theorem 5.4, we obtain the result that any Λ-tree can be embedded in a metrically homogeneous Λ-tree. However, by Proposition 6.1, the degree of the points in $\widehat{\mathbf{X}}$ will, in many cases, exceed the degree of any point in $\mathbf{X}$. Thus, our construction will not yield Theorem 2.3 in [**6**] for $\mathbb{R}$-trees. Moreover, the $\mathbb{R}$-trees T_α constructed there are complete, and by contrast we have the following result.

PROPOSITION 6.2. *In Theorem* 5.4, *if* $\Lambda = \mathbb{R}$ *and* $\mathbf{X}$ *has more than one G-orbit, then* $\widehat{\mathbf{X}}$ *is not complete as a metric space.*

Proof. Define a sequence $(g_n)_{n \geqslant 0}$ of elements of $\hat{G}$ recursively as follows. Put $g_0 = [(b_0)]$; if g_n has been defined and $g_n = [(x_0, i_1, \dots, i_n, x_n)]$ in reduced form, and x_n is in the G-orbit of b_j, choose $i_{n+1} \in I$ such that $i_{n+1} \neq j$, and choose $x_{n+1} \in X$ such that $0 < d(b_{i_{n+1}}, x_{n+1}) < 1/(n+1)^2$, then put

$$g_{n+1} = [(x_0, i_1, \dots, i_n, x_n, i_{n+1}, x_{n+1})],$$

which is in reduced form.

Then for $n > m$,

$$\hat{d}(g_m b, g_n b) = L((g_m^{-1} g_n) = \sum_{r=m+1}^{n} d(b_{i_r}, x_r) < \frac{1}{(m+1)^2} + \cdots + \frac{1}{n^2},$$

hence $(g_n b)_{n \geqslant 0}$ is a Cauchy sequence in $\widehat{\mathbf{X}}$.

Suppose $(g_n b)_{n \geqslant 0}$ converges; since the action of $\hat{G}$ is transitive, it converges to gb for some $g \in \hat{G}$, say $g = [(y_0, j_1, \dots, j_m, y_m)]$ in reduced form. Let k be maximal subject to

$$(x_0, i_1, x_1, \dots, i_k) = (y_0, j_1, y_1, \dots, j_k)$$

and apply Lemma 4.2 to the reduced words for g_n and g, where $n > m$. We obtain

$$\begin{aligned}
\hat{d}(g_n b, gb) &= L(g_n^{-1} g) = L(g_n) + L(g) - 2c(g_n, g) \\
&= \sum_{r=k}^{m} d(b_{j_r}, y_r) + \sum_{r=k}^{n} d(b_{i_r}, x_r) - 2(x_k \cdot y_k)_{b_{i_k}} \\
&\geqslant d(b_{j_k}, y_k) + d(b_{i_k}, x_k) - 2(x_k \cdot y_k)_{b_{i_k}} + d(b_{i_{m+1}}, x_{m+1}) \\
&= d(x_k, y_k) + d(b_{i_{m+1}}, x_{m+1}) \\
&\geqslant d(b_{i_{m+1}}, x_{m+1}) > 0,
\end{aligned}$$

for all $n > m$. This contradicts $g_n b \longrightarrow gb$ as $n \longrightarrow \infty$, hence $(g_n b)_{n \geqslant 0}$ is a Cauchy sequence which does not converge, as required.

If, in Theorem 5.4, G acts transitively on $\mathbf{X}$, then $G \longrightarrow \hat{G}$, given by $g \mapsto [(gb_0)]$ is a group isomorphism, and $\mathbf{X} \longrightarrow \widehat{\mathbf{X}}$, given by $gb_0 \mapsto [(gb_0)]b$ is a metric isomorphism. Thus, if $\Lambda = \mathbb{R}$, in this case $\widehat{\mathbf{X}}$ is complete if, and only if, $\mathbf{X}$ is.

7. *The group $\mathcal{RF}(G)$ and its associated $\mathbb{R}$-tree $\mathbf{X}_G$*

In recent joint work, the present authors have introduced a new construction, which associates to each (discrete) group G a group $\mathcal{RF}(G)$ together with a canonical $\mathbb{R}$-tree action $\mathcal{RF}(G) \to \mathrm{Isom}(\mathbf{X}_G)$; cf. [**3**]. To some extent, in particular when working with hyperbolic elements, these groups $\mathcal{RF}(G)$ appear as continuous analogues of free groups, whereas in other respects they behave more like amalgamated products, while in fact being neither. For the benefit of the reader, and since [**3**] has not yet appeared in print, we briefly review here the definition of the group $\mathcal{RF}(G)$ and its associated $\mathbb{R}$-tree $\mathbf{X}_G$.

Given a group G, let $\mathcal{F}(G)$ be the set of all functions $f\colon [0,\alpha] \to G$ defined on some closed real interval $[0,\alpha]$ with $\alpha \geqslant 0$. The real number α will be called the *length* of the function f, denoted $L(f)$. The *formal inverse* f^{-1} of an element $f \in \mathcal{F}(G)$ is the function defined on the same interval $[0,\alpha]$ as f by

$$f^{-1}(\xi) = f(\alpha-\xi)^{-1}, \quad 0 \leqslant \xi \leqslant \alpha.$$

We have $(f^{-1})^{-1} = f$. A function $f \in \mathcal{F}(G)$ is *reduced*, if to every interior point ξ_0 in the domain of f with $f(\xi_0) = 1_G$ and every real number ε satisfying $0 < \varepsilon \leqslant \min\{\alpha-\xi_0, \xi_0\}$ there exists δ such that $0 < \delta \leqslant \varepsilon$ and $f(\xi_0+\delta) \neq (f(\xi_0-\delta))^{-1}$. Clearly, every element in $\mathcal{F}(G)$ of length 0 is reduced; and if f is reduced, then so is its formal inverse f^{-1}. We denote by $\mathcal{RF}(G)$ the set of all reduced functions in $\mathcal{F}(G)$.

We now proceed to define a multiplication on $\mathcal{F}(G)$. Given $f, g \in \mathcal{F}(G)$ of lengths α, β respectively, let

$$\varepsilon_0 = \varepsilon_0(f,g) := \begin{cases} \sup \mathcal{E}(f,g), & f(\alpha) = \big(g(0)\big)^{-1} \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\mathcal{E}(f,g) := \Big\{\varepsilon \in [0, \min\{\alpha,\beta\}] : \ f(\alpha-\delta) = \big(g(\delta)\big)^{-1} \text{ for all } \delta \in [0,\varepsilon]\Big\},$$

and define fg on the interval $[0, \alpha+\beta-2\varepsilon_0]$ by

$$(fg)(\xi) := \begin{cases} f(\xi), & 0 \leqslant \xi < \alpha-\varepsilon_0 \\ f(\alpha-\varepsilon_0)g(\varepsilon_0), & \xi = \alpha-\varepsilon_0 \\ g(\xi-\alpha+2\varepsilon_0), & \alpha-\varepsilon_0 < \xi \leqslant \alpha+\beta-2\varepsilon_0. \end{cases}$$

One can show that the product of two reduced functions is again reduced, so that the above multiplication restricts to a binary operation on $\mathcal{RF}(G)$. Denote by $\mathbf{1}_G$ the function of length 0 with $\mathbf{1}_G(0) = 1_G$. It is easy to see that, for $f \in \mathcal{F}(G)$,

$$\mathbf{1}_G f = f = f\mathbf{1}_G$$

and

$$ff^{-1} = \mathbf{1}_G = f^{-1}f,$$

which shows in particular that $\mathbf{1}_G$ is a neutral element for $\mathcal{RF}(G)$ with the above multiplication, and that the formal inverse f^{-1} of an element $f \in \mathcal{RF}(G)$ is its inverse. Moreover, one can show that our multiplication is associative on $\mathcal{RF}(G)$, although the proof of this, as given in [**3**, Chapter 1], is surprisingly hard; hence, $\mathcal{RF}(G)$ when equipped with the multiplication defined above is a group. We note that the group G we started from is embedded into $\mathcal{RF}(G)$ as the subgroup

$$G_0 = \{ f \in \mathcal{RF}(G) : L(f) = 0 \}.$$

Further, it is not hard to see that the map L: $\mathcal{RF}(G) \to \mathbb{R}$ associating with each reduced function f the length $L(f)$ of its domain is a (real) Lyndon length function. This yields (by Theorem 1.1(ii)) the existence of an $\mathbb{R}$-tree $\mathbf{X}_G = (X_G, d_G)$ on which $\mathcal{RF}(G)$ acts, with a canonical basepoint x_0 such that $L = L_{x_0}$. In particular, the stabilizer $\mathrm{stab}_{\mathcal{RF}(G)}(x_0)$ of the point x_0 under the action of $\mathcal{RF}(G)$ is given by

$$\mathrm{stab}_{\mathcal{RF}(G)}(x_0) = G_0;$$

in particular, the action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$ is not free, whenever G is non-trivial. One can show that $\mathbf{X}_G$ is metrically complete, and that the action of $\mathcal{RF}(G)$ on $\mathbf{X}_G$ is transitive (see [**3**, Sections 2.2 and 2.4]).

8. *Universality of $\mathcal{RF}$-groups and their associated $\mathbb{R}$-trees*

Let G be a group acting freely and transitively on an $\mathbb{R}$-tree $\mathbf{X} = (X, d)$, and choose a basepoint $y_0 \in X$. Following [**1**], define an equivalence relation $\approx$ on $G - \{1_G\}$ by

$$g \approx h :\Longleftrightarrow c(g^{-1}, h^{-1}) = \tfrac{1}{2}\{L_{y_0}(g^{-1}) + L_{y_0}(h^{-1}) - L_{y_0}(gh^{-1})\} > 0$$

(transitivity of $\approx$ follows from the fact that $(c(g,h), c(h,k), c(g,k))$ is an isosceles triple for all $g, h, k \in G$; see the definition after [**2**, Chapter 1, Lemma 2.6]). Let s_g denote the equivalence class of g, set $s_{1_G} := \{1_G\}$, and let $S = \{s_g : g \in G\}$. Endow S with a group structure (which is arbitrary and need not be related to the structure of G), and denote the resulting group by H.

In what follows, we shall use some notation concerning Λ-trees which can be found in [**2**, Section 1, Chapter 2].

For $g \in G$, we define F_g: $[0, L_{y_0}(g)] \longrightarrow H$ as follows. Let ξ be such that $0 \leqslant \xi \leqslant L_{y_0}(g)$; the point in the segment $[y_0, gy_0]$ at distance ξ from y_0 has the form $g_\xi y_0$ for some unique $g_\xi \in G$ (since G acts freely and transitively), and we let

$$F_g(\xi) := s_{g_\xi}^{-1} s_{g^{-1}g_\xi}, \quad (g \in G,\ \xi \in [0, L_{y_0}(g)]).$$

By definition, $F_g \in \mathcal{F}(H)$ and $L(F_g) = L_{y_0}(g)$.

LEMMA 8.1. *For each $g \in G$, we have $F_g \in \mathcal{RF}(H)$.*

Proof. This is clear if $g = 1_G$, so we may assume that $g \neq 1_G$. It is enough to show that $F_g(\xi) \neq 1_H$ for every $\xi \in [0, L_{y_0}(g)]$. Fix such a value of ξ, and let $h = g_\xi$; we have to show that $s_h \neq s_{g^{-1}h}$. This is clear if $h = 1_G$ (i.e., if $\xi = 0$), so we may assume that $h \neq 1_G$. Now, the desired assertion is clear if $h = g$ (i.e., if $\xi = L_{y_0}(g)$), so we may assume further that $h \neq g$. Since $hy_0 \in [y_0, gy_0]$, we have $y_0 \in [h^{-1}y_0, h^{-1}gy_0]$, so

$$d(h^{-1}y_0, y_0) + d(y_0, h^{-1}gy_0) = d(h^{-1}y_0, h^{-1}gy_0) = d(y_0, gy_0);$$

that is,

$$L_{y_0}(h^{-1}) + L_{y_0}(h^{-1}g) = L_{y_0}(g).$$

It follows that $c(h^{-1}, h^{-1}g) = 0$, so $h \not\approx g^{-1}h$, as required.

LEMMA 8.2. *Let $g, h \in G$. For $0 \leqslant \xi < c(g,h)$, we have $F_g(\xi) = F_h(\xi)$; but $F_g(c(g,h)) \neq F_h(c(g,h))$, unless $g = h$.*

Proof. In the notation of [**2**, Chapter 2, Lemma 1.2], let $Y(y_0, gy_0, hy_0) = ky_0$, where $k \in G$, and suppose that $0 \leqslant \xi \leqslant c(g,h)$. Then

$$[y_0, gy_0] = [y_0, g_\xi y_0, ky_0, gy_0],$$

hence

$$[g_\xi^{-1} y_0, g_\xi^{-1} gy_0] = [g_\xi^{-1} y_0, y_0, g_\xi^{-1} ky_0, g_\xi^{-1} gy_0].$$

Therefore

$$\begin{aligned} d(y_0, g_\xi^{-1} gy_0) &= d(y_0, g_\xi^{-1} ky_0) + d(g_\xi^{-1} ky_0, g_\xi^{-1} gy_0) \\ &= d(y_0, g_\xi^{-1} ky_0) + d(y_0, k^{-1} gy_0); \end{aligned}$$

that is,

$$L_{y_0}(g_\xi^{-1} g) = L_{y_0}(g_\xi^{-1} k) + L_{y_0}(k^{-1} g),$$

and, consequently,

$$c(g_\xi^{-1} k, g_\xi^{-1} g) = \frac{1}{2} \left\{ L_{y_0}(g_\xi^{-1} k) + L_{y_0}(g_\xi^{-1} g) - L_{y_0}(k^{-1} g) \right\} = L_{y_0}(g_\xi^{-1} k).$$

Also, $g_\xi y_0 = h_\xi y_0$, both being the point on $[y_0, ky_0]$ at distance ξ from y_0, so $g_\xi = h_\xi$.

If $\xi < c(g,h)$, then $g_\xi y_0 \neq ky_0$, so $g_\xi \neq k$, and hence

$$c(g_\xi^{-1} k, g_\xi^{-1} g) = L_{y_0}(g_\xi^{-1} k) > 0,$$

so $s_{g^{-1} g_\xi} = s_{k^{-1} g_\xi}$. Symmetrically, interchanging g and h, we find that $s_{h^{-1} h_\xi} = s_{k^{-1} h_\xi}$; and, since $g_\xi = h_\xi$, it follows that $s_{g^{-1} g_\xi} = s_{h^{-1} h_\xi}$. Hence, we find that

$$F_g(\xi) = s_{g_\xi}^{-1} s_{g^{-1} g_\xi} = s_{h_\xi}^{-1} s_{h^{-1} h_\xi} = F_h(\xi), \quad 0 \leqslant \xi < c(g,h),$$

as claimed.

If $\xi = c(g,h)$, then $g_\xi = k = h_\xi$, and we have to show that, if $g \neq h$, then $s_{g^{-1}k} \neq s_{h^{-1}k}$. Since $g^{-1}k$, $h^{-1}k$ cannot both be equal to 1_G, it is enough to show that $c(k^{-1}g, k^{-1}h) = 0$, which in turn is equivalent to the assertion that

$$L_{y_0}(g^{-1} h) = L_{y_0}(k^{-1} g) + L_{y_0}(k^{-1} h). \tag{8·1}$$

Now $[gy_0, hy_0] = [gy_0, ky_0, hy_0]$, implying $[k^{-1} gy_0, k^{-1} hy_0] = [k^{-1} gy_0, y_0, k^{-1} hy_0]$. It follows that

$$d(y_0, g^{-1} hy_0) = d(k^{-1} gy_0, k^{-1} hy_0) = d(k^{-1} gy_0, y_0) + d(y_0, k^{-1} hy_0),$$

whence (8·1).

LEMMA 8.3. *For each $g \in G$, we have $F_{g^{-1}} = F_g^{-1}$.*

Proof. We note that

$$L(F_{g^{-1}}) = L_{y_0}(g) = L(F_g^{-1}).$$

Further, since, for $0 \leqslant \xi \leqslant L_{y_0}(g)$, $[y_0, gy_0] = [y_0, g_\xi y_0, gy_0]$ by definition of g_ξ, we have

$$[g^{-1} y_0, y_0] = [g^{-1} y_0, g^{-1} g_\xi y_0, y_0]$$

as well as $d(g^{-1} y_0, g^{-1} g_\xi y_0) = d(y_0, g_\xi y_0) = \xi$.

Hence,

$$d(y_0, g^{-1} g_\xi y_0) = d(y_0, g^{-1} y_0) - \xi = L_{y_0}(g) - \xi,$$

so that $g^{-1}g_\xi y_0$ is the unique point of the segment $[y_0, g^{-1}y_0]$ at distance $L_{y_0}(g) - \xi$ from y_0. By definition,

$$F_{g^{-1}}(L(F_g) - \xi) = F_{g^{-1}}(L_{y_0}(g) - \xi) = s_{g^{-1}g_\xi}\, s_{g_\xi}^{-1} = (F_g(\xi))^{-1},$$

and the lemma follows.

COROLLARY 8.4. *For all $g, h \in G$, we have $\varepsilon_0(F_g, F_h) = c(g^{-1}, h)$.*

Proof. If $g^{-1} = h$, then $c(g^{-1}, h) = L_{y_0}(g) = L_{y_0}(h)$. On the other hand, by Lemma 8.3,

$$F_g(L(F_g) - \xi)F_h(\xi) = F_g(L(F_g) - \xi)(F_g(L(F_g) - \xi))^{-1} = 1_G, \quad 0 \leqslant \xi \leqslant L(F_g);$$

hence,

$$\varepsilon_0(F_g, F_h) = \sup \mathcal{E}(F_g, F_h) = L(F_g) = L_{y_0}(g),$$

as desired.

Now assume that $g^{-1} \neq h$. By Lemma 8.2, we have

$$F_{g^{-1}}(\xi) = F_h(\xi), \quad 0 \leqslant \xi < c(g^{-1}, h),$$

$$F_{g^{-1}}(\xi) \neq F_h(\xi), \quad \xi = c(g^{-1}, h).$$

By Lemma 8.3, this implies that

$$F_g(L(F_g) - \xi)F_h(\xi) = 1_G, \quad 0 \leqslant \xi < c(g^{-1}, h),$$

$$F_g(L(F_g) - \xi)F_h(\xi) \neq 1_G, \quad \xi = c(g^{-1}, h),$$

and the corollary follows.

LEMMA 8.5. *The mapping ψ: $G \longrightarrow \mathcal{RF}(H)$ given by $g \mapsto F_g$ is an injective group homomorphism.*

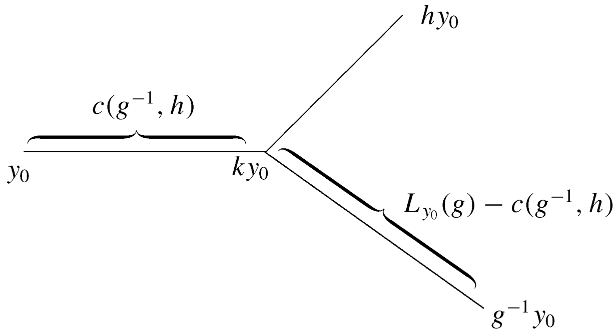
Proof. By Corollary 8.4, we have $c(g^{-1}, h) = \varepsilon_0(F_g, F_h)$, and ψ is length preserving by the definition of F_g, so that $L_{y_0}(g) = L(F_g)$ and, by definition of the function c,

$$\begin{aligned} L(F_{gh}) = L_{y_0}(gh) &= L_{y_0}(g) + L_{y_0}(h) - 2c(g^{-1}, h) \\ &= L(F_g) + L(F_h) - 2\varepsilon_0(F_g, F_h) = L(F_g\, F_h). \end{aligned}$$

Now, let $g, h \in G$, and let $Y(y_0, g^{-1}y_0, hy_0) = ky_0$. Then $c(g^{-1}, h) = d(y_0, ky_0) = L_{y_0}(k)$. Moreover, we have $gky_0 = Y(gy_0, y_0, ghy_0)$, so

$$c(g, gh) = d(y_0, gky_0) = d(g^{-1}y_0, ky_0) = d(g^{-1}y_0, y_0) - d(y_0, ky_0) = L_{y_0}(g) - c(g^{-1}, h),$$

since $ky_0 \in [y_0, g^{-1}y_0]$. This is illustrated by the following picture, and its translate by g.



By Lemma 8.2,

$$F_{gh}(\xi) = F_g(\xi), \quad 0 \leqslant \xi < L_{y_0}(g) - c(g^{-1}, h). \tag{8·2}$$

Next, suppose that $\xi = L_{y_0}(g) - c(g^{-1}, h)$, so that the point at distance ξ from y_0 on $[y_0, ghy_0]$ and on $[y_0, gy_0]$ is gky_0. Thus, $(gh)_\xi = gk = g_\xi$, and so

$$F_{gh}(\xi) = s_{gk}^{-1}\, s_{h^{-1}k}$$
$$\text{as well as} \quad F_g(\xi) = s_{gk}^{-1}\, s_k.$$

Also, ky_0 is the point on $[y_0, hy_0]$ at distance $c(g^{-1}, h)$ from y_0, thus

$$F_h(c(g^{-1}, h)) = s_k^{-1}\, s_{h^{-1}k}.$$

From the last three equations, we conclude that

$$F_g(L_{y_0}(g) - c(g^{-1}, h))\, F_h(c(g^{-1}, h)) = F_{gh}(L_{y_0}(g) - c(g^{-1}, h)). \tag{8·3}$$

Finally, suppose that $L_{y_0}(g) - c(g^{-1}, h) < \xi \leqslant L_{y_0}(gh)$. Then the point p at distance ξ from y_0 on $[y_0, ghy_0]$ is at distance

$$\xi - d(y_0, gky_0) = \xi - L_{y_0}(g) + c(g^{-1}, h) > 0$$

from gky_0. Further, $[gy_0, ghy_0] = [gy_0, gky_0, p, ghy_0]$, so $[y_0, hy_0] = [y_0, ky_0, g^{-1}p, hy_0]$, and

$$\begin{aligned} d(y_0, g^{-1}p) &= d(y_0, ky_0) + d(ky_0, g^{-1}p) \\ &= c(g^{-1}, h) + d(gky_0, p) \\ &= \xi - L_{y_0}(g) + 2c(g^{-1}, h). \end{aligned}$$

Thus, setting $\xi' := \xi - L_{y_0}(g) + 2c(g^{-1}, h)$, we have $g^{-1}p = h_{\xi'}y_0$, hence $p = gh_{\xi'}y_0$, and so $(gh)_\xi = gh_{\xi'}$. It follows that $F_{gh}(\xi) = s_{gh_{\xi'}}^{-1}\, s_{h^{-1}h_{\xi'}}$. It is easy to see that $ky_0 = Y(y_0, h_{\xi'}y_0, g^{-1}y_0)$, so

$$h_{\xi'}^{-1}ky_0 = Y(h_{\xi'}^{-1}y_0, y_0, h_{\xi'}^{-1}g^{-1}y_0),$$

and hence $c(h_{\xi'}^{-1}, h_{\xi'}^{-1}g^{-1}) = d(y_0, h_{\xi'}^{-1}ky_0) = d(p, gky_0) > 0$. Since clearly $h_{\xi'}, gh_{\xi'} \neq 1_G$, it follows that $s_{gh_{\xi'}} = s_{h_{\xi'}}$, so $F_h(\xi') = s_{h_{\xi'}}^{-1}\, s_{h^{-1}h_{\xi'}} = F_{gh}(\xi)$. Summarizing, we have shown that

$$F_{gh}(\xi) = F_h(\xi - L_{y_0}(g) + 2c(g^{-1}, h)), \quad L_{y_0}(g) - c(g^{-1}, h) < \xi \leqslant L_{y_0}(gh). \tag{8·4}$$

Combining Equations (8·2), (8·3) and (8·4), we obtain, abbreviating $\varepsilon_0(F_g, F_h)$ to ε_0,

$$\begin{aligned} F_{gh}(\xi) &= \begin{cases} F_g(\xi), & 0 \leqslant \xi < L(F_g) - \varepsilon_0 \\ F_g(L(F_g) - \varepsilon_0)\, F_h(\varepsilon_0), & \xi = L(F_g) - \varepsilon_0 \\ F_h(\xi - L(F_g) + 2\varepsilon_0), & L(F_g) - \varepsilon_0 < \xi \leqslant L(F_{gh}) \end{cases} \\ &= (F_g\, F_h)(\xi), \end{aligned}$$

for $0 \leqslant \xi \leqslant L(F_{gh})$. Thus $F_{gh} = F_g\, F_h$, showing that ψ is a group homomorphism. Also, since ψ is length preserving, it has trivial kernel, so is injective.

We are now in a position to establish our second main result.

THEOREM 8.6. *Let G be a group acting freely and transitively on an $\mathbb{R}$-tree $\mathbf{X} = (X, d)$. Then there exist a group H, an injective group homomorphism ψ: $G \to \mathcal{RF}(H)$, and a G-equivariant isometry ν: $\mathbf{X} \to \mathbf{X}_H$.*

Proof. Given a basepoint $y_0 \in X$, we have already constructed a group H and an injective homomorphism ψ: $G \to \mathcal{RF}(H)$. Define a map ν: $\mathbf{X} \to \mathbf{X}_H$ by

$$\nu(gy_0) := \psi(g)x_0 = F_g x_0,$$

making use of the fact that the action of G is free and transitive. Then we have, for $g, h \in G$,

$$\begin{aligned} d(\nu(gy_0), \nu(hy_0)) &= d(F_g x_0, F_h x_0) = d(x_0, F_g^{-1} F_h x_0) \\ &= L(F_g^{-1} F_h) = L(F_{g^{-1}h}) = L_{y_0}(g^{-1}h) \\ &= d(gy_0, hy_0); \end{aligned}$$

that is, ν is an isometry. Also, ν is G-equivariant, since

$$\nu(hgy_0) = \psi(hg)x_0 = \psi(h)\psi(g)x_0 = \psi(h)\nu(gy_0),$$

and the proof of the theorem is complete.

Finally, combining Theorems 5.4 and 8.6, we obtain the following important result.

THEOREM 8.7. *Let G be a group acting freely on an $\mathbb{R}$-tree $\mathbf{X} = (X, d)$. Then there exist a group H, a group embedding χ: $G \to \mathcal{RF}(H)$, and a G-equivariant isometry λ: $\mathbf{X} \to \mathbf{X}_H$ containing the canonical basepoint x_0 in its image.*

It would be interesting to combine the work of Morgan and Shalen on free $\mathbb{R}$-tree actions of surface groups with the main results of this paper to explicitly exhibit surface groups embedded into $\mathcal{RF}$-groups; see [**8**], [**9**] and also [**11**].

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