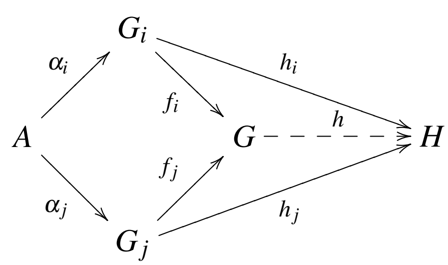
2. TWO BASIC CONSTRUCTIONS

Free Products with amalgamation. Let $\{G_i \mid i \in I\}$ be a family of gps, A a gp and $\alpha_i : A \longrightarrow G_i$ a monomorphism, $\forall i \in I$. A gp G is the *free product of the G_i with A amalgamated* (via the α_i) if $\exists$ homomorphisms $f_i : G_i \longrightarrow G$ such that $f_i\alpha_i = f_j\alpha_j \; \forall \; i, j \in I$, and if $h_i : G_i \longrightarrow H$ are homomorphisms with $h_i\alpha_i = h_j\alpha_j$ for all $i, j \in I$, then there is a unique homomorphism $h : G \longrightarrow H$ such that $hf_i = h_i$ for all $i \in I$.

Call h an *extension* of the homoms. h_i.



Note. Follows: G is generated by $\bigcup_{i\in I} f_i(G_i)$.
Let G_0 be the subgroup of G generated by this set.
Take $H = G_0$ and $h_i = f_i$, viewed as a mapping to G_0, in the defn.
$\exists$ an extn to a homomorphism $f : G \longrightarrow G_0$.
Let $e : G_0 \longrightarrow G$ be the inclusion map. Then ef and $\mathrm{id}_G : G \longrightarrow G$ are both extns of the maps $f_i : G_i \longrightarrow G$. Since the extn is unique, $ef = \mathrm{id}_G$, so e is onto, hence $G = G_0$.

Uniqueness. Let $f_i' : G_i \longrightarrow G'$ be homomorphisms such that $f_i'\alpha_i = f_j'\alpha_j$ for all $i, j \in I$, and if $h_i : G_i \longrightarrow H$ are homomorphisms with $h_i\alpha_i = h_j\alpha_j$ for all $i, j \in I$, then there is a unique homomorphism $h' : G' \longrightarrow H$ such that $h'f_i' = h_i$ for all $i \in I$.

In the defn, take $h_i = f_i'$, to obtain a homomorphism $f : G \longrightarrow G'$ such that $ff_i = f_i'$ for all i. Interchanging G and G', get $f' : G' \longrightarrow G$ such that $f'f_i' = f_i$ for all i. Take $H = G$, $h_i = f_i$ in the defn; both $f'f$ and $\mathrm{id}_G : G \longrightarrow G$ are extensions of the $f_i : G_i \longrightarrow G$. Since the extension is unique, $f'f = \mathrm{id}_G$. Interchanging G and G', $ff' = \mathrm{id}_{G'}$, so f and f' are inverse isomorphisms.

Existence. let $\langle X_i \mid R_i \rangle$ be a presn of G_i, with $X_i \cap X_j = \emptyset$ for $i \neq j$. Let Y be a set of generators for A and, for each $y \in Y$, $i \in I$ let $a_{i,y}$ be a word in $X_i^{\pm 1}$ representing $\alpha_i(y)$. Let

$$S = \left\{ a_{i,y}\, a_{j,y}^{-1} \mid y \in Y,\; i, j \in I,\; i \neq j \right\}.$$

Let G be the group with presentation $\langle \bigcup_{i\in I} X_i \mid \bigcup_{i\in I} R_i \cup S \rangle$. Then G is the free product with amalgamation. There is an obvious mapping $X_i \longrightarrow G$, which (by defn of a presentation) induces a homomorphism $f_i : G_i \longrightarrow G$. The existence and uniqueness of the mapping h in the definition follows by defn of a presentation.

If G is the free product of $\{G_i \mid i \in I\}$ with A amalgamated, write $G = \ast_A G_i$.
In case of two groups $\{G_1, G_2\}$, write $G = G_1 \ast_A G_2$.

Example. let $G_1 = \langle x \rangle$, $G_2 = \langle y \rangle$, $A = \langle a \rangle$ inf. cyclic.
$\alpha_1 : a \mapsto x^2$, $\alpha_2 : a \mapsto y^3$.
$G = G_1 \ast_A G_2$ has presentation $\langle x, y \mid x^2 = y^3 \rangle$, after Tietze transformations.

Structure of free prods with amalgamation.

Let $G = \ast_A G_i$ via $\alpha_i : A \to G_i$, $i \in I$. To represent elts of G by certain words, identify A with $\alpha_i(A)$ and assume $G_i \cap G_j = A$ for $i \neq j$.

Define a *word* to be a finite sequence

$$w = (g_1, \ldots, g_k),$$

where $k \geq 1$ and $g_1 \in G_{i_1}, \ldots, g_k \in G_{i_k}$ for some $i_1, \ldots, i_k \in I$.
Elt. of G *represented* by w is $f_{i_1}(g_1) \ldots f_{i_k}(g_k)$.
Since $\bigcup_{i \in I} f_i(G_i)$ generates G and the f_i are homoms, every element of G is repd by some word.

Definition. Let $w = (g_1, \ldots, g_k)$ be a word with $g_j \in G_{i_j}$, $1 \leq j \leq k$. Then w is *reduced* if

(1) $i_j \neq i_{j+1}$ for $1 \leq j \leq k-1$;
(2) $g_j \notin A$ for $1 \leq j \leq k$, unless $k = 1$.

If (1) fails, can replace w by $(g_1, \ldots, g_j g_{j+1}, \ldots, g_k)$, representing the same element of G.
If (2) fails, can also replace w by a shorter word.
Hence every elt of G is represented by a reduced word (take a word of shortest length representing it).

For every $i \in I$, choose a transversal T_i for the cosets $\{Ax \mid x \in G_i\}$, with $1 \in T_i$. Let $g \in G$ be represented by the reduced word $(g_1, \ldots, g_k)$ with $g_j \in G_{i_j}$.

Can write:

$$\begin{aligned}
g_k &= a_k r_k && (a_k \in A,\ r_k \in T_{i_k}) \\
g_{k-1} a_k &= a_{k-1} r_{k-1} && (a_{k-1} \in A,\ r_{k-1} \in T_{i_{k-1}}) \\
&\vdots && \vdots \\
g_1 a_2 &= a_1 r_1 && (a_1 \in A,\ r_1 \in T_{i_1}).
\end{aligned}$$

Then $g_1 = a_1 r_1 a_2^{-1}$, $g_2 = a_2 r_2 a_3^{-1}, \ldots, g_k = a_k r_k$ and $(a_1, r_1, r_2, \ldots, r_k)$ is a word representing g.

Definition. A *normal* word is a word $(a, r_1, \ldots, r_k)$ where $a \in A$, $k \geq 0$, $r_j \in T_{i_j} \setminus \{1\}$ for some $i_j \in I$ $(1 \leq j \leq k)$, and $i_j \neq i_{j+1}$ for $1 \leq j \leq k-1$.

Thus any element of G is represented by a normal word.
($a \in A$ is represented by the normal word (a).)

Theorem 2.1 (Normal Form Theorem). *Any element of G is represented by a unique normal word.*

Proof. Need to show uniqueness. Let $W =$ set of normal words.
Will define an action of G on W, equivalently, a homom. $h : G \longrightarrow S(W)$ (symmetric group on W).
By defn of G, suffices to define, for $i \in I$, a homom. $h_i : G_i \longrightarrow S(W)$ with $h_i \alpha_i = h_j \alpha_j$ for all i, $j \in I$.
So with the identifications made, need homomorphisms h_i which agree on A.

Let $W_i = \{(1, r_1, \ldots, r_k) \in W \mid r_1 \notin T_i\} \quad (i \in I)$.
Define $\theta_i : G_i \times W_i \longrightarrow W$ as follows. If $g \in G_i$, write $g = ar$, where $r \in T_i$, $a \in A$. Then

$$\theta_i(g, (1, r_1, \ldots, r_k)) = \begin{cases} (a, r, r_1, \ldots, r_k) & \text{if } r \neq 1 \text{ (i.e. } g \notin A) \\ (a, r_1, \ldots, r_k) & \text{if } r = 1. \end{cases}$$

θ_i is bijective (exercise).

Now G_i acts on $G_i \times W_i$ by left mult. on the first co-ordinate, giving a homomorphism $\lambda_i : G_i \longrightarrow S(G_i \times W_i)$, where $\lambda_i(g)(x, w) = (gx, w)$.
Set $h_i(g) = \theta_i \lambda_i(g) \theta_i^{-1}$. Since λ_i is a homomorphism, so is h_i.
If $a \in A$,

$$h_i(a)(a', r_1, \ldots, r_k) = (aa', r_1, \ldots, r_k)$$

and RHS is independent of i.
Thus the h_i define a homomorphism $h : G \longrightarrow S(W)$.

Let $g \in G$ be represented by the normal word $w = (a, r_1, \ldots, r_k)$. Then $h(g)(1) = w$ (induction on k) so w is uniquely determined by g. □

Corollary 2.2. (1) *The homomorphisms f_i are injective;*
(2) *no reduced word of length greater than* 1 *represents the identity element of G.*

Proof. (1) If $f_i(g) = 1$, write $g = ar$ with $a \in A$, $r \in T_i$. Then the normal word (a, r) (or (a), if $r = 1$) represents 1, so $a = r = 1$, ie $g = 1$.

(2) If a reduced word has length $k > 1$, procedure above gives a normal word of length $k+1$ representing same elt of G, which cannot be 1 by Theorem 2.1. □

By Cor. 2.2(1), the f_i can be suppressed.

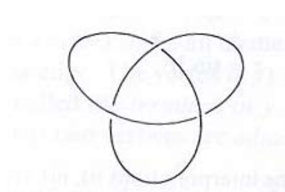
Important special case: $A = \{1\}$.
G is called the *free product* of the family $\{G_i \mid i \in I\}$, written $G = \ast_{i\in I} G_i$ (or $G = G_1 * G_2$ for two groups).
A reduced word is a word $(g_1, \dots, g_k)$ such that $g_j \in G_{i_j}$, $g_j \neq 1$ unless $k = 1$, and $i_j \neq i_{j+1}$ for $1 \leq j \leq k-1$.
Normal Form Theorem simplifies: every element of G is represented by a unique reduced word. Also, the G_i embed in G.
Defining property simplifies to: given any collection of homomorphisms $h_i : G_i \longrightarrow H$, there is a unique extension to a homomorphism $h : G \longrightarrow H$.
Presn used to show existence simplifies.
Let $\langle X_i \mid R_i\rangle$ be a presentation of G_i (via some mapping which will be suppressed), with $X_i \cap X_j = \emptyset$ for $i \neq j$. Then $\ast_{i\in I} G_i$ has presentation $\langle \bigcup_{i\in I} X_i \mid \bigcup_{i\in I} R_i\rangle$.
[This is obtained from the presentation above by taking the empty presentation of the trivial group, with no generators and no relations.]

Examples. (1) $\langle x, y \mid x^2 = 1,\ y^2 = 1\rangle$ is a presentation of $C_2 * C_2$, the *infinite dihedral group*. Has D_n as a homomorphic image.
(2) $C_2 * C_3$ has presentation $\langle x, y \mid x^2 = 1,\ y^3 = 1\rangle$. This group is called the *modular group*, and is isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$.
(3) F free with basis $\{x_i \mid i \in I\}$; then $F = \ast_{i\in I}\langle x_i\rangle$, a free product of infinite cyclic groups.

(4) Trefoil knot K; group of the knot, $\pi_1(\mathbb{R}^3 - K)$ has presn

$$\langle a, b \mid aba = bab\rangle$$

$\xrightarrow[\text{transf.}]{\text{Tietze}}$ $\langle x, y \mid x^2 = y^3\rangle$ (previous example).



HNN-extensions. Suppose B, C are subgroups of a group A and $\gamma : B \longrightarrow C$ is an isomorphism.

Let G have presentation $\langle \{t\} \cup X \mid R_1 \cup R_2 \rangle$, where

$$\begin{aligned} X &= \left\{x_g \mid g \in A\right\}, \quad t \notin X \\ R_1 &= \left\{x_g x_h x_{gh}^{-1} \mid g,h \in A\right\} \\ R_2 &= \left\{t x_b t^{-1} x_{\gamma(b)}^{-1} \mid b \in B\right\}. \end{aligned}$$

$\exists$ a homom. $f : A \to G$ given by $a \mapsto x_a$; will show f is injective.

Definition. The group G is called an HNN-extension with *base* A, *associated pair* of subgroups B, C and *stable letter* t.

Presn is often abbreviated to $\langle t, A \mid \mathrm{rel}(A), tBt^{-1} = \gamma(B)\rangle$ or just

$$\langle t, A \mid \mathrm{rel}(A), tBt^{-1} = C\rangle.$$

More generally, Let $\langle Y \mid S\rangle$ be a presentation of A, let $\left\{b_j \mid j \in J\right\}$ be a set of words in $Y^{\pm 1}$ representing a set of generators for B, and let c_j be a word representing $\gamma(b_j)$ [more accurately, γ(the generator represented by b_j)]. Then

$$\langle \{t\} \cup Y \mid S \cup \left\{t b_j t^{-1} = c_j \mid j \in J\right\}\rangle$$

is also a presentation of G. (Exercise.)

Example. (Baumslag-Solitar) $\langle x, y \mid xy^2x^{-1} = y^3\rangle$ is an HNN-extension, base an infinite cyclic group $A = \langle y\rangle$, stable letter x, assoc.pair $\langle y^2\rangle$, $\langle y^3\rangle$.

(Famous example of a non-Hopfian group.)

In general, G is generated by $f(A) \cup \{t\}$, so every element of G is represented in an obvious way by a word

$$(g_0, t^{e_1}, g_1, t^{e_2}, g_2, \ldots, t^{e_k}, g_k)$$

where $k \geq 0$, $e_i = \pm 1$ and $g_i \in A$ for $0 \leq i \leq k$.

Definition. Such a word is *reduced* if it has no part of the form t, b, t^{-1} with $b \in B$ or t^{-1}, c, t with $c \in C$.

Any element of G is represented by a reduced word (take a word of minimal length representing it).

Choose a transversal T_B for the cosets $\{Bg \mid g \in A\}$ and a transversal T_C for the cosets $\{Cg \mid g \in A\}$, with $1 \in T_B$, T_C.

Definition. A *normal word* is a reduced word $(g_0, t^{e_1}, r_1, t^{e_2}, r_2, \ldots, t^{e_k}, r_k)$, where $g_0 \in A$, $r_i \in T_B$ if $e_i = 1$ and $r_i \in T_C$ if $e_i = -1$.

Suppose $(g_0, t^{e_1}, g_1, t^{e_2}, g_2, \ldots, t^{e_k}, g_k)$ is a reduced word with $k \geq 1$. If $e_k = 1$, write $g_k = br$ with $b \in B$, $r \in T_B$. Then

$$(g_0, t^{e_1}, g_1, t^{e_2}, g_2, \ldots, g'_{k-1}, t^{e_k}, r)$$

where $g'_{k-1} = g_{k-1}\gamma(b)$, represents the same element of G.
If $e_k = -1$, write $g_k = cr$ with $c \in C$ and $r \in T_C$. Then

$$(g_0, t^{e_1}, g_1, t^{e_2}, g_2, \ldots, g'_{k-1}, t^{e_k}, r)$$

where $g'_{k-1} = g_{k-1}\gamma^{-1}(c)$, represents the same element of G.

Repetition leads to a normal word representing the same element of G.

Theorem 2.3. (Normal Form Theorem.) *Every element of G is represented by a unique normal word.*

Proof. Omitted. □

Corollary 2.4. (1) *The homomorphism $f : A \longrightarrow G$ is injective;*
(2) (Britton's Lemma) *no reduced word $(g_0, t^{e_1}, g_1, t^{e_2}, g_2, \ldots, t^{e_k}, g_k)$ with $k > 0$ represents the identity element of G.* □

More generally, given A and a family $\gamma_i : B_i \longrightarrow C_i$ $(i \in I)$ of isomorphisms, where B_i, C_i are subgroups of A, we can form the HNN-extension with presentation (in abbreviated form)

$$\langle t_i \ (i \in I),\ G \mid \mathrm{rel}(G),\ t_i B_i t_i^{-1} = \gamma_i(B_i) \ (i \in I)\rangle.$$

There are generalisations of the Normal Form Theorem and its corollary.

An application.

Theorem 2.5. *Any countable group can be embedded in a 2-generator group.*

Proof. Let $G = \{g_0, g_1, g_2, \ldots\}$, where $g_0 = 1$.
Let F be free with basis $\{a, b\}$.
$\langle b^n a b^{-n} \ (n \geq 0)\rangle$ is a free subgp of F with the given elts as basis. Similarly $\langle a^n b a^{-n} \ (n \geq 0)\rangle$ is a free subgp. (See Exercise 2, Sheet 1.)
Claim: in $G * F$, $\langle g_n a^n b a^{-n} \ (n \geq 0)\rangle$ is free with basis the given elts. For the homoms. $G \to F$, $g \mapsto 1$ and $\mathrm{id}_F : F \to F$ have an extn to a homom. $f : G * F \to F$. If w is repd by a non-empty reduced word in $\{g_n a^n b a^{-n} \mid (n \geq 0)\}^{\pm 1}$ of positive length, then $f(w)$ is repd by a corresponding reduced word in $\{a^n b a^{-n} \mid (n \geq 0)\}^{\pm 1}$, so $f(w) \neq 1$, hence $w \neq 1$. Claim follows (see Exercise 1, Sheet 1).
Can form HNN-extn

$$H = \langle t, G * F \mid \mathrm{rel}(G * F), t(b^n a b^{-n})t^{-1} = g_n a^n b a^{-n} \ (n \geq 0)\rangle.$$

Then $G \leq G * F \leq H$, $g_n \in \langle t, a, b\rangle$, and $(n = 0)$ $tat^{-1} = b$, so $H = \langle t, a\rangle$. □