

## Topology Summary

This is a summary of the results discussed in lectures. These notes will be gradually growing as the time passes by. Most of the time proofs will not be given. You are supposed to be able to understand them from your notes. Parts in red are things that were not mentioned in the corresponding lecture but may be used in the future lectures or in exercise sheets. They will be discussed when needed but it would be very useful if you could think about them before that.

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### Lecture 1-1,2

**Definition.** Sets  $A$  and  $B$  have the same cardinality if there is an invertible function  $f : A \mapsto B$ .

We write  $\text{Card}(A) = \text{Card}(B)$ .

*Remark.* In more simple but less precise terms  $\text{Card}(A) = \text{Card}(B)$  if there is a one-to-one correspondence between the elements of  $A$  and  $B$ .

**Definition.** A set  $A$  is countable if it has the same cardinality as the set of natural numbers  $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ .

We proved the following statement:

**Theorem 1.** The sets  $[0, 1]$  and  $(0, 1]$  have the same cardinality.

The proof makes use of a simple observation that  $A = \{1, 2, \dots\}$  and  $B = \{2, 3, \dots\}$  have the same cardinality.

**Definition.** If  $X$  is a set and  $\mathcal{T}$  is a family of subsets of  $X$  then  $\mathcal{T}$  is a topology on  $X$  if it satisfies the following properties:

1. if  $U_1, U_2 \in \mathcal{T}$  then  $U_1 \cap U_2 \in \mathcal{T}$ ,
2. if  $U_\alpha : \alpha \in \Gamma$  with  $U_\alpha \in \mathcal{T}$  for each  $\alpha \in \Gamma$  then  $\bigcup_{i \in \Gamma} U_i \in \mathcal{T}$ , where  $\Gamma$  is a set of indices.
3.  $\emptyset, X \in \mathcal{T}$

A topological space is a set  $X$  together with a topology  $\mathcal{T}$ . We call the sets in  $\mathcal{T}$  open sets.

In the future we often say that  $X$  is a topological space without mentioning the relevant topology  $\mathcal{T}$ ; it would always be clear from the context which topology is the “relevant” one. One sometimes says that  $X$  is equipped with topology  $\mathcal{T}$ .

*Remark.* Any finite intersection of open sets is open (prove this). You cannot do infinite intersections. For example in  $\mathbb{R}$  the sets  $U_n = (-1/n, 1/n)$ ,  $1 \leq n < \infty$ , are all open but  $\bigcap_n U_n = \{0\}$  which is not open.

**Definition.** A set  $A$  in a topological space  $X$  is closed if  $X \setminus A$  (i.e., the complement of  $A$ ) is open.

*Examples.*

1. Suppose  $X$  is a set. Let  $\mathcal{T}$  be the power set of  $X$ ; i.e.,  $\mathcal{T}$  contains all subsets of  $X$ ; i.e., every set is open. Trivially it satisfies the axioms for a topology. We call this the discrete topology.
2. Let  $X$  be a metric space. Let  $\mathcal{T}$  be the collection of metric-open sets in  $X$ ; that is the topology-open sets are exactly the metric-open sets. The axioms are satisfied (indeed that is why we chose our axioms for a topological space).  
Remark: The discrete metric gives the discrete topology.
3. Suppose  $X$  is a set. Let  $\mathcal{T} = \{\emptyset, X\}$ . It is easy to check that this is a topology. We call this the indiscrete topology.
4. Suppose  $X$  is an infinite set. A set  $U$  is open if either  $U = \emptyset$  or  $X \setminus U$  only contains finitely many points. Check that if  $\mathcal{T}$  is a collection of such sets then it is a topology. We call this the co-finite topology.

Sometimes we can have two different metrics on the same set. These can sometimes give different topologies e.g.,  $\mathbb{R}$  with the normal metric and  $\mathbb{R}$  with the discrete metric. But sometimes different metrics give the same topology: e.g.,  $\mathbb{R}^2$  with Euclidean metric and  $\mathbb{R}^2$  with  $l_1$  (Manhattan metric).

**Definition.** Suppose that  $d_1$  and  $d_2$  are two metrics on the same space  $X$ . We say they are equivalent if every  $d_1$ -open ball  $B_{d_1}(x, r)$  contains a  $d_2$ -open ball  $B_{d_2}(x, s)$  and vice versa.

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## Lecture 2-1,2

**Definition.** Suppose  $X$  is a topological space. We say  $X$  is Hausdorff if for every pair of distinct points  $x, y \in X$  we can find disjoint open sets  $U$  and  $V$  with  $x \in U$  and  $y \in V$ .

*Exercise.* Every metric space is Hausdorff.

**Definition.** Suppose that  $X$  and  $Y$  are topological spaces and  $f: X \mapsto Y$  is a function. We say  $f$  is continuous if for every open set  $V$  in  $Y$  we have  $f^{-1}(V)$  is open in  $X$ .

*Remarks.*

1.  $f^{-1}(V)$  is defined to be  $\{x \in X : f(x) \in V\}$
2. This is not saying the image of an open set is open: it is saying the pre-image of an open set is open.
3. The pre-image is better behaved than the image: e.g.  $f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2)$  and similarly for unions. This would not be true for images.

*Exercise.* Suppose  $X$  and  $Y$  are topological spaces and  $f: X \mapsto Y$ .  $f$  is continuous if and only if for every closed set  $V \subset Y$  the set  $f^{-1}(V)$  is closed.

*Exercise.* Suppose that  $X$  and  $Y$  are metric spaces and  $f: X \mapsto Y$  is a function. We have two definitions of continuity: one using the metric and one using the corresponding topology (i.e., one using  $\varepsilon, \delta$  and one using the pre-image of open sets). Show that these two definitions are the same: i.e.,  $f$  is continuous in the metric sense if and only if it is continuous in the topology sense.

**Theorem 2.** Suppose  $X, Y$  and  $Z$  are topological spaces and  $f: X \mapsto Y$  and  $g: Y \mapsto Z$  are (both) continuous. Then  $g \circ f: X \mapsto Z$  is continuous.

**Definition.** Suppose  $X$  and  $Y$  are topological spaces. A function  $f: X \rightarrow Y$  is a homeomorphism if

1.  $f$  is a bijection *and*
2.  $f$  is continuous *and*
3.  $f^{-1}$  is also continuous.

*Remark.* We really do need all three of these conditions.

**Definition.** If  $f: X \rightarrow Y$  is a homeomorphism then the topological spaces  $X$  and  $Y$  are homeomorphic.

*Examples.* Suppose  $f: [0, 1] \rightarrow \mathbb{R}$  is continuous. We know  $f$  is bounded. But this is not true on  $(0, 1)$ . So  $[0, 1]$  and  $(0, 1)$  are not homeomorphic.

*Examples.* Are  $\mathbb{R}$  and  $\mathbb{R}^2$  homeomorphic? Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}^2$  is a homeomorphism. Then  $f$  maps  $\mathbb{R} \setminus \{0\}$  to  $\mathbb{R}^2 \setminus \{f(0)\}$ . The first looks like two pieces: the second like one piece. This suggests they are not homeomorphic (and this will be formalized later).

*Examples.* Are  $[0, 1)$  and  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$  homeomorphic? Consider the function  $f: [0, 1) \rightarrow S^1$  by  $f(x) = e^{2\pi i x}$ . Is this a homeomorphism? It is a bijection; it is continuous. BUT the inverse is not continuous! It is not a homeomorphism.

*Exercise.* Show that homeomorphism is an equivalence relation.

*Remark.* We did this in the lecture.

**Definition.** A topological property is one which is preserved under homeomorphism. That is if  $X$  and  $Y$  are homeomorphic then  $X$  has the property if and only if  $Y$  has the property.

Essentially we are only interested in topological properties.

**Theorem 3.** Being Hausdorff is a topological property.

*Exercise.* Prove this theorem.

*Remark.* In fact you should prove a somewhat stronger statement: if  $f: X \rightarrow Y$  is a continuous injection and  $Y$  is Hausdorff then  $X$  is Hausdorff.

*Examples.* For metric spaces we have a notion of boundedness: that is a metric space is bounded if there is some real number  $M$  such that  $d(x, y) \leq M$  for all  $x, y$ . Boundedness is not a topological property. For example,  $(0, 1)$  and  $(1, \infty)$  are homeomorphic but one is bounded and one is not.

## Sequences in Topological Spaces

**Definition.** Suppose that  $X$  is a topological space and  $(x_n)_{n=1}^{\infty}$  is a sequence of points in  $X$ . We say that  $x_n$  converges to  $y$  if for any  $U$  open in  $X$  with  $y \in U$  there is  $N$  such that for all  $n > N : x_n \in U$ .

## Sequences in Topological Spaces

Warning: sequences are not as well behaved in topological spaces as in metric spaces. Here is once again the definition of a converging sequence.

**Definition.** Suppose that  $X$  is a topological space and  $(x_n)_{n=1}^{\infty}$  is a sequence of points in  $X$ . We say that  $x_n$  converges to  $x$  if

$$\forall (U \text{ open in } X \text{ with } x \in U) \exists N \forall n > N : x_n \in U.$$

*Examples.* 1. If  $X$  has discrete topology and  $x_n$  converges to  $x$ . Take  $U = \{x\}$ . Then there is a  $N$  such that for all  $n > N$  we have  $x_n \in \{x\}$ , i.e.,  $x_n = x$ . The only sequences which converge are those which are ‘eventually  $x$ ’: i.e., there is an  $N$  when  $x_n$  reaches  $x$  and stays there for evermore.

2. What if  $X$  has indiscrete topology. Then there is only one open set  $U$  containing  $x$ . It is trivially true that there is an  $N$  such that  $x_n \in U$  for all  $n > N$ ; indeed  $N = 1$  will do. Every sequence converges to every single point (all at the same time)!

**Theorem 4.** Suppose  $X$  is a Hausdorff topological space and  $(x_n)_{n=1}^{\infty}$  is a sequence and  $x_n \rightarrow x$  and  $x_n \rightarrow y$  then  $x = y$ .

Informally this tells us that convergent sequences have unique limits.

**Theorem 5.** Suppose  $X$  and  $Y$  are topological spaces and  $f: X \rightarrow Y$  is continuous and  $(x_n)_{n=1}^{\infty}$  is a sequence of points in  $X$  with  $x_n \rightarrow x$ . Then  $f(x_n) \rightarrow f(x)$ .

**Corollary 6.** Convergence is a topological property.

*Remark.* Completeness is not a topological property.

**Definition.** Suppose  $A$  is a subset of a topological space  $X$ . A point  $x \in X$  is a limit point of  $A$  if every open set  $U \ni x$  meets  $A$  at a point other than  $x$  itself.

**Theorem 7.** Suppose  $A$  is a subset of a topological space  $X$ . Then  $A$  is closed if and only if  $A$  contains all of its limit points.

## Subspaces

**Definition.** Suppose  $(X, \mathcal{T})$  is a topological space and  $S$  is a subset of  $X$ . Then the subspace topology on  $S$  is the family

$$\mathcal{T}_S = \{U \cap S : U \in \mathcal{T}\},$$

that is the open sets in  $S$  are the intersection of open sets  $U$  in  $X$  with  $S$ .

**Definition.** If  $S$  is a subspace of a topological space  $X$  the inclusion map is the function  $i: S \rightarrow X$  given by  $i(x) = x$ .

**Lemma 8.** The inclusion map  $i: S \rightarrow X$  is continuous.

**Lemma 9.** Suppose  $X$  and  $Y$  are topological spaces, that  $S$  is a subspace of  $X$  and that  $f: Y \rightarrow X$  is a function. Then  $f$  is continuous and if and only if  $i \circ f$  is continuous.

## Paths and Path Connectedness.

**Definition.** Suppose  $X$  is a topological space. Then a path in  $X$  is a continuous function  $\alpha: [0, 1] \rightarrow X$ . It starts at  $\alpha(0)$  and finishes at  $\alpha(1)$ .

**Definition.** A topological space  $X$  is path connected if for any two points  $x, y \in X$  there is a path starting at  $x$  and finishing at  $y$  (abbreviated to ‘a path from  $x$  to  $y$ ’).

**Theorem 10.** Suppose  $X$  and  $Y$  are topological spaces and  $X$  is path connected and  $f: X \rightarrow Y$  is continuous. Then  $f(X)$  is path connected.

**Corollary 11.** Path connectedness is a topological property.

*Remark.* Deleting a point idea. E.g.  $(0, 1]$  and  $(0, 1)$  are not homeomorphic.

**Definition** (Concatenation of paths.). Suppose that  $X$  is a topological space and that  $\alpha, \beta: [0, 1] \rightarrow X$  are paths, and that  $\alpha(1) = \beta(0)$  (i.e.  $\beta$  starts where  $\alpha$  finishes. Then the join path  $\alpha \cdot \beta: [0, 1] \rightarrow X$  is given by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & t \leq 1/2 \\ \beta(2t - 1) & t \geq 1/2. \end{cases}$$

[The two definitions of  $\alpha \cdot \beta(1/2)$  agree so that is ok.]

**Lemma 12** (The Gluing Lemma.). Suppose that  $X$  and  $Y$  are topological spaces and  $A, B$  are closed subsets of  $X$  with  $A \cup B = X$ . Further suppose that  $f_A: A \rightarrow Y$  and  $f_B: B \rightarrow Y$  are continuous and  $f_A$  and  $f_B$  agree on  $A \cap B$ . Then there is a (unique) continuous function  $f: X \rightarrow Y$  which agrees with  $f_A$  on  $A$  and  $f_B$  on  $B$ .

*Proof.* 1. The only (and the obvious) way to define  $f: X \rightarrow Y$  is to set  $f(x) = f_A(x)$  if  $x \in A \setminus B$ ,  $f(x) = f_B(x)$  if  $x \in B \setminus A$ , and  $f(x) = f_A(x) = f_B(x)$  if  $x \in A \cap B$ .

2. Note that for any set  $U \subset Y$  one has  $f^{-1}(U) = f_A^{-1}(U) \cup f_B^{-1}(U)$ . If  $U$  is closed, then  $f_A^{-1}(U)$  and  $f_B^{-1}(U)$  are closed subsets in  $A$  and  $B$ , respectively because of the continuity of  $f_A$  and  $f_B$ . It remains to note that closed subsets of  $(A, \mathcal{T}_A)$  and of  $(B, \mathcal{T}_B)$  are closed in  $(X, \mathcal{T}_X)$ . Hence  $f^{-1}(U)$  is closed in  $(X, \mathcal{T}_X)$ , which means that  $f$  is continuous.  $\square$

*Remark 1.*  $f$  is continuous iff for any closed  $U \subset Y$  the pre-image  $f^{-1}(U)$  is closed (see ex. 1 in Exercise sheet 4).

**Corollary.** Join paths are continuous (i.e., join paths are paths).

*Exercise.* Explain how this statement follows from Lemma 12.

**Theorem 13.** Suppose  $X$  is a topological space and let  $\sim$  be the relation on  $X$  defined by  $x \sim y$  if there is an  $xy$  path. The  $\sim$  is an equivalence relation.

**Definition.** Each equivalence class in the relation  $\sim$  is called a path component.

*Exercise.* Show that the number of path components is a topological property.

## Compactness

**Definition.** Suppose that  $(X, \mathcal{T})$  is a topological space and  $A$  is a subset of  $X$ . Then a family of open subsets  $C \subset \mathcal{T}$  is a cover of  $A$  if for any  $x \in A$  there is a  $U \in C$  such that  $x \in U$ .

Compactness: one of the most important properties in topology.

**Definition.** Suppose that  $X$  is a topological space and  $A$  is a subset of  $X$ . Then  $A$  is compact if every open cover of  $A$  has a finite sub-cover.

*Examples:* 1)  $[0, 1]$  is a compact set; 2) any closed subset of  $[0, 1]$  is compact; 3) any closed bounded subset of  $\mathbb{R}^n$  is compact.

**Definition.** A topological space  $X$  is sequentially compact if every sequence  $(x_n)_{n=1}^{\infty}$  of points in  $X$  has a convergent subsequence (to something in  $X$ ).

**Theorem 14.** I. Suppose  $X$  is a compact metric space. Then  $X$  is sequentially compact.  
II. If a metric space  $X$  is sequentially compact then it is compact.

The detailed proof of this theorem was given in the lecture. Here we summarize the three steps leading to the proof of part II:

1. Suppose  $X$  is sequentially compact and  $\varepsilon > 0$ . Then there exist finitely many points  $x_1, x_2, \dots, x_n$  such that every point of  $X$  is within  $\varepsilon$ -distance of one of the  $x_i$  (i.e., the balls  $B(x_i, \varepsilon)$  cover  $X$ ).
2. Suppose that  $C$  is a cover of  $X$ . If it does not have a finite sub-cover then for any  $\varepsilon > 0$  there is at least one ball  $B(y, \varepsilon)$  which cannot be covered by a finite subcover of  $C$ .
3. Consider now a sequence  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$  and let  $B(y_j, \varepsilon_j)$  be a sequence of balls such that each of them cannot be covered by a finite number of open sets from  $C$ . Due to sequential compactness, we can choose a subsequence  $y_{j_k} \rightarrow y$  as  $k \rightarrow \infty$ . Since there is a  $U \in C$  s.t.  $y \in U$ , there is a ball  $B(y, \bar{\varepsilon}) \subset U$ . But  $B(y_{j_k}, \varepsilon_{j_k}) \subset B(y, \bar{\varepsilon})$  whenever  $\rho(y_{j_k}, y) < \frac{1}{3}\bar{\varepsilon}$  and  $\varepsilon_{j_k} < \frac{1}{3}\bar{\varepsilon}$  which contradicts the assumption that all  $B(y_{j_k}, \varepsilon_{j_k})$  cannot be covered by a finite number of subsets from  $C$ .

**Theorem 15** (Heine-Borel Lemma). The set  $[0, 1]$  is compact (in  $\mathbb{R}$ ).

**Lemma 16.** If  $X$  is a Hausdorff space then  $x \in X$  is a closed set.

**Theorem 17.** Suppose  $X$  is a Hausdorff topological space and that  $A$  is a compact subset of  $X$ . Then  $A$  is closed.

**Theorem 18.** If  $A$  is a closed subset of a compact set then  $A$  is compact.

**Theorem 19.** Suppose that  $X$  is a metric space and that  $A \subset X$  is compact. Then  $A$  is bounded.

## Properties of Functions on Compact Spaces

**Theorem 20.** Suppose  $X$  and  $Y$  are topological spaces and  $X$  is compact and  $f: X \rightarrow Y$  is continuous. Then  $f(X)$ , the image of  $X$ , is compact. In particular compactness is a topological property.

**Corollary 21.** Suppose  $X$  is a compact topological space and that  $f: X \rightarrow \mathbb{R}$  is continuous. Then  $f$  is bounded.

**Lemma 22.** Suppose that  $A \subset \mathbb{R}$  is not closed. Then it is not compact.

*Remark.* If  $X$  is a compact topological space and  $f: X \rightarrow \mathbb{R}$  is continuous. Then  $f(X)$  is compact: in particular  $f(X)$  is closed and bounded. So for example it attains its least upper bound.

**Corollary 23.** A set  $A \subset \mathbb{R}$  is compact if and only if it is closed and bounded.

**Theorem 24.** Suppose that  $X, Y$  are topological spaces,  $X$  is compact and  $Y$  is Hausdorff. Further suppose that  $f: X \rightarrow Y$  is a continuous bijection. Then  $f$  is a homeomorphism.

**Lemma 25 (Lebesgue's Covering Lemma).** Suppose that  $X$  is a compact metric space and  $U_i, i \in I$  is an open cover. Then there exists  $\varepsilon > 0$  such that for every  $x \in X$  there exists  $i \in I$  with  $B(x, \varepsilon) \subset U_i$ .

Exercise. Prove this Lemma.

## Product Spaces

**Definition.** Suppose  $X$  and  $Y$  are topological spaces. Then a set  $W \subset X \times Y$  is open in the product topology if it is a union of sets of the form  $U \times V$  with  $U$  open in  $X$  and  $V$  open in  $Y$ . More formally  $W = \bigcup_{i \in I} U_i \times V_i$  with each  $U_i$  open in  $X$  and each  $V_i$  open in  $Y$ .

**Lemma 26.** A set  $W \subset X \times Y$  is open if and only if for every point  $w \in W$  there exists  $U$  open in  $X$  and  $V$  open in  $Y$  with  $U \times V \subset W$  and  $w \in U \times V$ .

Compare with

**Lemma.** A set  $W$  in a metric space is open if and only if for every point  $w \in W$  there exists  $\varepsilon > 0$  such that  $B(w, \varepsilon) \subset W$ .

*Remark.* In a metric space we mostly care about open balls BUT there are many other open sets. In a product topology we mostly care about box like sets  $U \times V$  BUT there are many other open sets.

**Lemma 27.** The product topology is a topology.

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**Lecture 7-1,2**

**Definition.** Suppose  $X \times Y$  is a product space. We define the projection maps  $P_X: X \times Y \rightarrow X$  by  $P_X((x, y)) = x$ . Similarly define  $P_Y: X \times Y \rightarrow Y$  by  $P_Y((x, y)) = y$ .

**Lemma 28.** The projection maps  $P_X$  and  $P_Y$  are continuous.

**Lemma 29.** Suppose  $X, Y$  and  $Z$  are topological spaces and that  $f: Z \rightarrow X \times Y$ . Then  $f$  is continuous if and only if both  $P_X \circ f: Z \rightarrow X$  and  $P_Y \circ f: Z \rightarrow Y$  are continuous.

**Theorem 30.** The topological space  $X \times Y$  is compact if and only if  $X$  and  $Y$  are compact topological spaces.

*Proof.* I. Set  $W \equiv X \times Y$ . Suppose  $W$  is compact. Then  $X = P_X(W)$  and is compact because  $P_X$  is continuous.  $Y$  is dealt with similarly.

II. Suppose that  $X$  and  $Y$  are compact. The proof of compactness of  $W$  requires several steps.

1. Let  $C = \{H_i, i \in \Gamma\}$  be a cover of  $W \equiv X \times Y$ . Then, by the definition of a cover, for every  $(x, y) \in W$  there is  $H_i \in C$  s. t.  $(x, y) \in H_i$ . Next, by the definition of an open set, there are  $U_{xy} \in \mathcal{T}_X, V_{xy} \in \mathcal{T}_Y$  such that  $(x, y) \in U_{xy} \times V_{xy} \in H_i$ . The collection of open subsets  $\bar{C} = \{U_{xy} \times V_{xy}, (x, y) \in W\}$  forms a cover of  $W$ . If  $\bar{C}$  contains a finite subcover, say  $\{U_j \times V_j, j = 1, \dots, n\}$  then the corresponding  $\{H_j : H_j \supset U_j \times V_j, j = 1, \dots, n\}$  is a finite subcover of  $W$  formed from the elements of  $C$ .

It thus suffices to prove the statement for covers consisting of "simple" open sets. From now on we consider only this case:

$$C = \{H_i : H_i = U_i \times V_i, i \in \Gamma, U_i \in \mathcal{T}_X, V_i \in \mathcal{T}_Y\}.$$

2. Sets of the form  $U \times Y$ , where  $U \in \mathcal{T}_X$  will be called tubes.

**Lemma 31.** (The Tube Lemma.) For every  $x \in X$  there is an open set  $U_x \in \mathcal{T}_X$  containing  $x$  and a finite collection of subsets from  $C$  which cover the tube  $U_x \times Y$

*Proof.* For  $x \in X$  let  $C_x$  be the collection of all sets  $U_i \times V_i$  from  $C$  such that  $x \in U_i$ . We emphasize the dependence of these sets on  $x$  by writing

$$C_x = \{U_i^{(x)} \times V_i^{(x)} : x \in U_i^{(x)} \text{ and } U_i^{(x)} \times V_i^{(x)} \in C\}$$

Note that  $\{V_i^{(x)} : x \in U_i^{(x)} \text{ and } U_i^{(x)} \times V_i^{(x)} \in C\}$  form a cover of  $Y$  (prove this!). Since  $Y$  is compact, it has a finite subcover, say  $D = \{V_j^{(x)}, 1 \leq j \leq n(x)\}$ .

Now set  $U_x = \bigcap_{j=1}^{n(x)} U_j^{(x)}$ , where  $U_j^{(x)}$  are the sets related to those in  $D$  in a natural way:  $\{U_j^{(x)} \times V_j^{(x)} \in C, V_j^{(x)} \in D\}$ .  $U_x$  is an open set (as a finite intersection of open sets) and

$$U_x \times Y \subset \bigcup_{j=1}^{n(x)} U_x \times V_j^{(x)} \subset \bigcup_{j=1}^{n(x)} U_j^{(x)} \times V_j^{(x)}$$

which means that  $\{U_j^{(x)} \times V_j^{(x)}, 1 \leq j \leq n(x)\}$  is a finite cover of the tube  $U_x \times Y$  by the sets from  $C$ . □



3. The collection  $\{U_x, x \in X\}$  forms a cover of  $X$  which has a finite subcover, say  $U_{x_k}, 1 \leq k \leq m$ . Obviously

$$X \times Y \subset \bigcup_{k=1}^m U_{x_k} \times Y$$

which means that the tubes  $U_{x_k} \times Y$  cover  $W$ . But by step 2, each tube is covered by a finite number of sets from  $C$ . Hence there is also a finite cover of  $W$  (namely, by the sets  $\{U_{x_k}^j \times V_{x_k}^j, 1 \leq k \leq m, 1 \leq j \leq n(x_k)\}$ ).  $\square$

**Corollary 32.** Suppose  $A$  is a subset of  $\mathbb{R}^n$ . Then  $A$  is compact if and only if  $A$  is closed and bounded.

## Quotient Spaces

(These are also called identification spaces.)

**Definition.** Suppose  $X$  is a topological space and  $\sim$  is an equivalence relation on  $X$ . The space  $X/\sim$  is the set of equivalence classes of  $\sim$  with the following topology. Define the quotient map  $q: X \rightarrow X/\sim$  by  $x$  goes to the equivalence class containing  $x$ . A set  $U \subset X/\sim$  is open in  $X/\sim$  if  $q^{-1}(U)$  is open in  $X$ .

*Remark.* If  $V$  is open in  $X$  then  $q(V)$  might NOT be open.

*Remark.* Every open set in  $X/\sim$  is the image under  $q$  of an open set. Indeed, suppose that  $U$  is open in  $X/\sim$  then  $q(q^{-1}(U)) = U$  since  $q$  is surjective, and  $q^{-1}(U)$  is open. However, *not* every image of an open set is open. You may wish to construct an example with  $V$  open in  $X$  but  $q(V)$  is not open in  $X/\sim$ .

**Lemma 33.** The quotient map  $q: X \rightarrow X/\sim$  is continuous.

*Remark.* The quotient topology was chosen to make Lemma 33 true; indeed it is the topology with the most open sets making  $q$  continuous.

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## Lecture 8-1,2

**Notation.** Commutative diagrams. (See the relevant picture in your notes.)

**Theorem 34.** Suppose  $X$  and  $Y$  are topological spaces and  $\sim$  is an equivalence relation on  $X$ . Further suppose that the diagram  $f, \hat{f}, q$  commutes (that is  $f = \hat{f} \circ q$ ). Then  $f$  is continuous if and only if  $\hat{f}$  is continuous.

*Remark.* Suppose  $f: X \rightarrow Y$ . When can we find an  $\hat{f}$  such that this diagram commutes? We need  $f$  to satisfy  $f(x) = f(y)$  whenever  $x \sim y$ . And indeed, if  $f$  satisfied this we can find an  $\hat{f}$ : just define  $\hat{f}([x]) = f(x)$  and it does not matter which representative of  $[x]$  we choose.

**Definition.** We say  $f: X \rightarrow Y$  respects  $\sim$  if  $f(x) = f(y)$  whenever  $x \sim y$ .

**Notation** If  $X$  is a topological space with an equivalence relation  $\sim$  then by  $[x]$  we denote the equivalence class of  $x$ .

*Examples.* Let  $X = [0, 1]$  and  $X/\sim$  be the quotient space formed from  $X$  (with the usual metric topology) by identifying 0 and 1 (and no other points). Prove that  $X/\sim$  is homeomorphic to  $S = \{z \in \mathbb{C} : |z| = 1\}$ .

We considered examples where a cylinder, a Mobius strip, and a Klein bottle were obtained from a strip via imposition of the relevant equivalence relations on the strip (see notes for more details).

*Examples.* Fourier Series: functions from  $\mathbb{R}$  to  $\mathbb{R}$  which are periodic say period  $2\pi$  or functions from  $\mathbb{R}/\sim$  where  $x \sim y$  if  $x - y \in 2\pi\mathbb{Z}$ . Since (as we shall see)  $\mathbb{R}/\sim$  is (homeomorphic to) the unit circle  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ . Usually the second of these is easier.

**Theorem 35.** The space  $\mathbb{R}/\mathbb{Z}$  is homeomorphic to  $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ .

*Exercise.* Prove this theorem

*Examples* (Warning!). The space  $\mathbb{R}/\mathbb{Q}$  that is  $\mathbb{R}$  quotiented by the equivalence relation  $\sim$  where  $x \sim y$  if  $x - y \in \mathbb{Q}$ . As we saw above every open set in  $\mathbb{R}/\sim$  is the image under  $q$  of an open set in  $\mathbb{R}$ . We show that the image on every non-empty open set in  $\mathbb{R}$  is all of  $\mathbb{R}/\mathbb{Q}$ . Thus  $\mathbb{R}/\mathbb{Q}$  has the indiscrete topology. It does not have a ‘nice topology’: it is not even Hausdorff. Quotient spaces can be nasty!

**Notation.** A few spaces that we will come across:

1.  $\mathbb{R}$  and  $\mathbb{R}^n$ ;
2. the circle  $S^1$  which is  $\{z \in \mathbb{C} : |z| = 1\}$  or  $\mathbb{R}/\mathbb{Z}$  (homeomorphic so the ‘same’ space). Alternatively we can view it as  $[0, 1]$  with 0 and 1 identified. Identified means ‘put these points in the same equivalence class’
3. the  $n$ -dimensional sphere  $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ . Note  $S^n$  lives in  $\mathbb{R}^{n+1}$ . The point is the sphere  $S^n$  is sort of  $n$  dimensional in itself: e.g., the sphere  $S^2$  is sort of 2-dimensional (think surface of the earth).
4. the  $n$ -dimensional closed ball  $B^n = \{x \in \mathbb{R}^{n+1} : \|x\| \leq 1\}$ .
5. the  $n$ -dimensional open ball  $D^n = \{x \in \mathbb{R}^{n+1} : \|x\| < 1\}$ .
6. the torus  $S^1 \times S^1$ .

## Paths and Homotopies

Recall a path is a continuous function  $\alpha: [0, 1] \rightarrow X$ . Note the image  $\alpha([0, 1]) \subset X$  and the path are not the same: we could go round a circle once or twice or backwards and these are different paths with the same image  $\alpha([0, 1]) \subset X$ . We also defined the join of two paths.

Is  $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ ? (Obviously we need  $\alpha(1) = \beta(0)$  and  $\beta(1) = \gamma(0)$ . The images are the same but the functions are not: e.g.,  $\alpha \cdot (\beta \cdot \gamma)(1/2) = \alpha(1)$  whereas  $(\alpha \cdot \beta) \cdot \gamma(1/2) = \gamma(0)$ . Indeed in the first case we do  $\alpha$  in time  $1/2$  and then  $\beta$  and  $\gamma$  each in time  $1/4$ ; whereas in the second case we do  $\alpha$  and  $\beta$  each in time  $1/4$  and then  $\gamma$  in time  $1/2$ . The paths are different.

This is annoying: we don't like non-associative operations. We want to get round this. We try and 'pretend' these two paths are actually the same.

**Definition.** Suppose that  $\alpha$  and  $\beta$  are two paths in  $X$  with  $\alpha(0) = \beta(0)$  and  $\alpha(1) = \beta(1)$ . We say  $\alpha$  and  $\beta$  are homotopic relative  $\{0, 1\}$  if there is a continuous function  $F: [0, 1] \times [0, 1] \rightarrow X$  satisfying the following conditions:

1.  $F(t, 0) = \alpha(t)$ ,
2.  $F(t, 1) = \beta(t)$ ,
3.  $F(0, s) = \alpha(0) = \beta(0)$  and
4.  $F(1, s) = \alpha(1) = \beta(1)$ .

We write  $\alpha \simeq \beta$ . We call  $F$  a homotopy.

Think of fixing the second coordinate equal to  $t$ : if  $t = 0$  we get the path  $\alpha$ , if  $t = 1$  we get the path  $\beta$ ; more generally for fixed  $t$  we have a path starting at  $\alpha(0)$  and finishing at  $\alpha(1)$ . You can think of this as continuously deforming the path  $\alpha$  to the path  $\beta$  while keeping the endpoints fixed.

**Definition.** A subset  $A$  of  $\mathbb{R}^n$  is convex if for any two points  $x, y \in A$  the line segment from  $x$  to  $y$  is in  $A$ . More formally the point  $(1 - t)x + ty \in A$  for all  $t \in [0, 1]$ .

**Lemma 36.** Suppose that  $A$  is a convex subset of  $\mathbb{R}^n$  and that  $\alpha$  and  $\beta$  are two paths in  $A$  that start at the same point and finish at the same point. Then  $\alpha \simeq \beta$ .

**Definition.** A loop is a path  $\alpha$  which starts and finishes at the same point; i.e.,  $\alpha(0) = \alpha(1)$ . We call this start/finish the base point of  $\alpha$ .

**Definition.** The constant path at  $x$  denoted  $c_x$  is the path given by  $c_x(t) = x$  for all  $t$ .

**Definition.** A loop  $\alpha$  is homotopic to a point (or is contractible) if there is a constant path  $c_x$  with  $\alpha \simeq c_x$ . (Of course if this is true  $x$  must be the basepoint of the loop.)

**Definition.** A topological space  $X$  is simply connected if every loop in  $X$  is homotopic to a point.

If  $\alpha$  is a loop based at  $x$  in a convex set  $A$  then  $\alpha \simeq c_x$ ; i.e,  $A$  is simply connected.

*Examples.* (1)  $\mathbb{R}^n$  is simply connected.

(2) If  $X = \{z : z \in \mathbb{C}, |z| = 1\}$  then the path  $\alpha(t) = e^{2\pi it}$ ,  $0 \leq t \leq 1$ , is not contractible (this has still to be proved). Hence the unit circle  $X$  is not a simply connected space.

(3) A unit sphere in  $\mathbb{R}^3$  is simply connected (we shall prove a more general statement).

(4)  $B^n$  and  $D^n$  are all simply connected since they are convex subsets of  $\mathbb{R}^{n+1}$ .

**Lemma 37.** If topological spaces  $X$  and  $Y$  are homeomorphic and  $X$  is simply connected then also  $Y$  is simply connected. (In other words, simply connectedness is a topological property).

*Proof.* Let  $f: X \rightarrow Y$  be the homeomorphism,  $\beta$  be a path in  $Y$ , and  $\alpha$  a path in  $X$  s. t.  $f \circ \alpha = \beta$ . If  $F$  is a homotopy relating  $\alpha$  to  $x = \alpha(0) = \alpha(1)$  then  $f \circ F$  is a homotopy relating  $\beta$  to  $y = \beta(0) = \beta(1)$  (check this statement!).  $\square$

*Remark.* Note that the continuous image of a simply connected space might not be simply connected.

*Remark.* The above proof in fact implies a slightly more general statement: if  $f: X \rightarrow Y$  is continuous then the images of homotopic paths are homotopic. More formally:

**Lemma 38.** Suppose that  $X$  and  $Y$  are topological spaces, that  $f: X \rightarrow Y$  is continuous, and that  $\alpha$  and  $\beta$  are paths in  $X$  with  $\alpha \simeq \beta$ . Then  $f \circ \alpha: [0, 1] \rightarrow Y$  and  $f \circ \beta: [0, 1] \rightarrow Y$  are homotopic relative  $\{0, 1\}$ .

*Exercise 1.* Prove this lemma.

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### Lecture 9-1, 2, 3

#### Useful examples of simply connected spaces.

**Theorem 39.** The  $n$ -dimensional sphere  $S^n$  is simply connected if  $n \geq 2$ .

**Lemma 40.**  $X = S^n \setminus \{(0, \dots, 0, 1)\}$  is homeomorphic to  $\mathbb{R}^n$ . In particular  $X$  is simply connected.

The proof of Theorem 39 follows now from the following general

**Lemma 41.** Suppose  $X$  is a path connected topological space with  $X = U \cup V$  with  $U$  and  $V$  open and simply connected and  $U \cap V$  non-empty and path connected. Then  $X$  is simply connected.

#### Continuous functions and functions continuous at a point.

In this subsection we prove one more criteria for continuity of a function. In some applications below, it may be easier to use Lemma 42 than the gluing lemma.

**Definition.** We say that a function  $f: X \rightarrow Y$  is continuous at a point  $x \in X$  if for every open set  $V \subset Y$  such that  $f(x) \in V$  there is an open set  $U \subset X$  such that  $x \in U$  and  $f(U) \subset V$ .

**Lemma 42.** Suppose that  $X, Y$  are topological spaces,  $f: X \rightarrow Y$  is a function. Then  $f$  is continuous if and only if it is continuous at each point  $x \in X$ .

*Proof.* 1. Suppose that  $f$  is continuous,  $x \in X$  is a point, and  $V \subset Y$  is open with  $f(x) \in V$ . Then  $f^{-1}(V)$  is an open set in  $X$  by the definition of continuity. Obviously,  $x \in f^{-1}(V)$  and  $f(f^{-1}(V)) = V$ . Hence  $f$  is continuous at each  $x \in X$ .

2. Suppose that  $f$  is continuous at each  $x \in X$ ,  $V \subset Y$  is open. We have to show that  $U \equiv f^{-1}(V)$  is an open set in  $X$ . Let  $x$  be a point in  $U$ . Then  $f(x) \in V$  and, by the definition of continuity at  $x$ , there is an open set  $U_x \subset X$  such that  $x \in U_x$  and  $f(U_x) \subset V$ .

But then  $U_x \subset U$  for any  $x \in U$  and therefore  $\bigcup_{x \in U} U_x \subset U$ . On the other hand,  $\bigcup_{x \in U} U_x$  contains all points from  $U$  because  $x \in U_x$ . Hence  $U = \bigcup_{x \in U} U_x$  and therefore  $U$  is open.  $\square$

## Properties of homotopies

**Lemma 43.** Homotopy is an equivalence relation on the set of all paths in a topological space  $X$ .

*Proof.* 0.  $\alpha \simeq \alpha$  because  $F(t, s) = \alpha(t)$  is a homotopy relating  $\alpha$  to  $\alpha$ .

1.  $\alpha \simeq \beta$  implies  $\beta \simeq \alpha$ . Indeed, if  $F_{\alpha, \beta}(t, s)$  relates  $\alpha$  to  $\beta$  then  $F_{\beta, \alpha}(t, s) := F_{\alpha, \beta}(t, 1 - s)$  relates  $\beta$  to  $\alpha$ .

2.  $\alpha \simeq \beta \simeq \gamma$  implies  $\beta \simeq \gamma$ . To see this, set

$$F_{\alpha, \gamma}(t, s) = \begin{cases} F_{\alpha, \beta}(t, 2s) & s \leq 1/2 \\ F_{\beta, \gamma}(t, 2s - 1) & s \geq 1/2. \end{cases}$$

(Check that  $F_{\alpha, \gamma}(t, s)$  can be defined in this way; note that continuity of  $F$  follows from either the gluing lemma or Lemma 42.)  $\square$

**Lemma 44.** Suppose that  $\alpha$  is a path starting at  $x$  and finishing at  $y$  then  $c_x \cdot \alpha \simeq \alpha$ . Similarly,  $\alpha \cdot c_y \simeq \alpha$ .

**Proof** By definition,

$$c_x \cdot \alpha(t) = \begin{cases} c_x(2t) = x & t \leq 1/2 \\ \alpha(2t - 1) & t \geq 1/2. \end{cases}$$

Consider a continuous function  $g: [0, 1] \rightarrow [0, 1]$  defined by  $g(t) = \begin{cases} 0 & t \leq 1/2 \\ 2t - 1 & t \geq 1/2. \end{cases}$  Note that

$c_x \cdot \alpha(t) = \alpha(g(t))$  Set  $F(t, s) = \alpha(st + (1 - s)g(t))$ . It remains to check that  $F$  is a homotopy relating the paths  $c_x \cdot \alpha$  and  $\alpha$ .  $\square$

*Exercise.* Write down the proof of  $\alpha \cdot c_y \simeq \alpha$ .

**Lemma 45.** Suppose that  $\alpha, \beta$  and  $\gamma$  are paths. Then  $\alpha \cdot (\beta \cdot \gamma) \simeq (\alpha \cdot \beta) \cdot \gamma$ .

**Proof** By definition,

$$(\alpha \cdot \beta) \cdot \gamma(t) = \begin{cases} \alpha(4t) & 0 \leq t \leq 1/4 \\ \beta(4t - 1) & 1/4 \leq t \leq 1/2 \\ \gamma(2t - 1) & 1/2 \leq t \leq 1, \end{cases} \text{ and } \alpha \cdot (\beta \cdot \gamma)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(4t - 2) & 1/2 \leq t \leq 3/4 \\ \gamma(4t - 3) & 3/4 \leq t \leq 1. \end{cases}$$

Consider a continuous function  $g: [0, 1] \rightarrow [0, 1]$  defined by  $g(t) = \begin{cases} 2t & 0 \leq t \leq 1/4 \\ t + 1/4 & 1/4 \leq t \leq 1/2 \\ t/2 + 1/2 & 1/2 \leq t \leq 1. \end{cases}$

Denote  $\delta(t) = \alpha \cdot (\beta \cdot \gamma)(t)$  and note that  $\delta(g(t)) = (\alpha \cdot \beta) \cdot \gamma(t)$ . Set  $F(t, s) = \delta(st + (1-s)g(t))$ . It remains to check that  $F$  is a homotopy relating the paths  $(\alpha \cdot \beta) \cdot \gamma$  and  $(\alpha \cdot \beta) \cdot \gamma$ .  $\square$

*Remark.* The function  $g$  is a piece-wise linear map s. t.  $g([0, 1/4]) = [0, 1/2]$ ,  $g([1/4, 1/2]) = [1/2, 3/4]$ ,  $g([1/2, 1]) = [3/4, 1]$ .

**Lemma 46.** Suppose  $\alpha \simeq \alpha'$  and  $\beta \simeq \beta'$  then  $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$ .

**Proof** Let  $F_1$  be a homotopy relating  $\alpha$  and  $\alpha'$  and  $F_2$  a homotopy relating  $\beta$  and  $\beta'$ . Set

$$F(t, s) = \begin{cases} F_1(2t, s) & t \leq 1/2 \\ F_2(2t - 1, s) & t \geq 1/2. \end{cases}$$

Then  $F: [0, 1] \times [0, 1] \rightarrow X$  is a continuous function (say by gluing lemma or by Lemma 42; check this statement!) and

1.  $F(t, 0) = \alpha \cdot \beta(t)$  because  $F(t, 0) = F_1(2t, 0) = \alpha(2t)$  if  $t \leq 1/2$  and  $F(t, 0) = F_2(2t - 1, 0) = \beta(2t - 1)$  if  $t \geq 1/2$  (we use here the definition of  $F_1, F_2$ ).

2. Similarly,  $F(t, 1) = \alpha' \cdot \beta'(t)$ .

3.  $F(0, s) = F_1(0, s) = \alpha(0) = \alpha'(0)$

4.  $F(1, s) = F_2(1, s) = \beta(1) = \beta'(1)$

We see that  $F$  is a homotopy relating  $\alpha \cdot \beta(t)$  and  $\alpha' \cdot \beta'(t)$  (which thus are homotopic).  $\square$

**Definition.** Suppose that  $\alpha$  is a path. Define the inverse path  $\alpha^{-1}$  by  $\alpha^{-1}(t) = \alpha(1 - t)$ . (NOTE: this is not the inverse function!)

**Lemma 47.** Suppose  $\alpha$  starts at  $x$  and finishes at  $y$ . Then  $\alpha \cdot \alpha^{-1} \simeq c_x$  and  $\alpha^{-1} \cdot \alpha \simeq c_y$ .

*Proof.* By definition,

$$\alpha \cdot \alpha^{-1}(t) = \begin{cases} \alpha(2t) = x & t \leq 1/2 \\ \alpha(2 - 2t) & t \geq 1/2. \end{cases}$$

Consider a continuous function  $g: [0, 1] \rightarrow [0, 1]$  defined by  $F(t, s) = \begin{cases} \alpha(2ts) & 0 \leq t \leq 1/2 \\ \alpha((2 - 2t)s) & 1/2 \leq t \leq 1. \end{cases}$

Note that  $F(t, 1) = \alpha \cdot \alpha^{-1}(t)$  and  $F(t, 0) = F(0, s) = F(1, s) = \alpha(0)$ . Hence  $F$  is a homotopy relating the paths  $c_x$  and  $\alpha \cdot \alpha^{-1}$ .  $\square$

**Lemma 48.** Suppose that  $X$  is a path connected space and that  $x \in X$ . Further suppose that all loops based at  $x$  are homotopic to a point (i.e., to  $c_x$ ) then every loop in  $X$  is homotopic to a point: i.e.,  $X$  is simply connected.

*Proof.* Let  $y \in X$ ,  $y \neq x$ . Let  $\beta$  be a loop based at  $y$  and  $\gamma$  be a path from  $x$  to  $y$ . Set  $\alpha = (\gamma \cdot \beta) \cdot \gamma^{-1}$ .

Note that  $\alpha$  is a loop based at  $x$  and therefore  $\alpha \simeq c_x$ . Hence

$$\gamma^{-1} \cdot \alpha \stackrel{\text{Lemma 46}}{\simeq} \gamma^{-1} \cdot c_x \stackrel{\text{Lemma 44}}{\simeq} \gamma^{-1}.$$

Hence, similarly,  $(\gamma^{-1} \cdot \alpha) \cdot \gamma \simeq \gamma^{-1} \cdot \gamma \simeq c_y$ . But  $(\gamma^{-1} \cdot \alpha) \cdot \gamma = (\gamma^{-1} \cdot ((\gamma \cdot \beta) \cdot \gamma^{-1})) \cdot \gamma \simeq \beta$  (use Lemma 45 to explain this!). Thus  $\beta \simeq c_y$ .  $\square$

## The Fundamental Group

**Definition.** Suppose  $X$  is a path connected topological space and  $x \in X$ . Let  $P_1(X, x)$  denote the collection of all loops in  $X$  based at  $x$ , with the join operation ‘ $\cdot$ ’

*Remark.* We can always define the join of two loops  $\alpha, \beta$  in  $P_1(X, x)$  since  $\alpha$  finishes where  $\beta$  starts.

*Remark.*  $P_1(X, x)$  is almost a group; that is it is a group if we work ‘up to homotopy’.

**Definition.** For  $\alpha$  in  $P_1(X, x)$  let  $[\alpha]$  denote the equivalence class of  $\alpha$  under homotopy; that is  $[\alpha] = \{\beta \in P_1(X, x) : \beta \simeq \alpha\}$ .

**Definition.** Let  $\pi_1(X, x)$  denote the collection of all equivalence classes in  $P_1(X, x)$  with the join operation ‘ $\cdot$ ’ defined by  $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$ . This is called the fundamental group of  $X$ .

*Remark.* The proof of the theorem below starts with checking that this join operation is well defined: that is that it respects the equivalence relation.

**Theorem 49.** The fundamental group is a group.

*Proof.* We shall use the following notations. If  $\alpha, \beta, \gamma$  are paths in  $X$  then by  $\alpha', \beta', \gamma'$  we denote paths from the corresponding homotopy classes:  $\alpha \simeq \alpha', \beta \simeq \beta', \gamma \simeq \gamma'$ . The following properties of homotopies will be used:

$$(1) c_x \cdot \alpha \simeq \alpha \cdot c_x \simeq \alpha$$

$$(2) \alpha \cdot \beta \simeq \alpha' \beta'$$

$$(3) \alpha \cdot \alpha^{-1} \simeq c_x$$

$$(4) (\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$$

We first show that  $[\alpha] \cdot [\beta] = [\alpha'] \cdot [\beta']$  which mean that this multiplication does not depend on the choice of the representative from the homotopy classes.

Indeed, by definition  $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$  and  $[\alpha'] \cdot [\beta'] = [\alpha' \cdot \beta']$ . By property (2)  $\alpha \cdot \beta \simeq \alpha' \beta'$  and therefore  $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$  since  $\simeq$  is an equivalence relation. Hence  $[\alpha] \cdot [\beta] = [\alpha'] \cdot [\beta']$ .

The group properties now follow easily.

1)  $[c_x]$  is the identity of the group:  $[\alpha] \cdot [c_x] = [\alpha \cdot c_x] = [\alpha]$  because of (1) and similarly  $[c_x] \cdot [\alpha] = [\alpha]$ .

2)  $[\alpha] \cdot [\alpha^{-1}] = [\alpha \cdot \alpha^{-1}] = [c_x]$  because of (3) and this means that  $[\alpha^{-1}]$  is the inverse to  $[\alpha]$ .

3) Finally, we prove associativity. By definition of the ‘ $\cdot$ ’ operation

$$([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha \cdot \beta] \cdot [\gamma] = [(\alpha \cdot \beta) \cdot \gamma]$$

and

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) = [\alpha] \cdot [\beta \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)].$$

By (4),  $[(\alpha \cdot \beta) \cdot \gamma] = [\alpha \cdot (\beta \cdot \gamma)]$ . Hence  $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$ .

This finishes the proof of the Theorem. □

*Remark.*  $X$  is simply connected if and only if  $\pi_1(X, x)$  is the trivial group (i.e., the group with just one element).

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**Lecture 10-1, 2, 3**

**Lemma 50.** Suppose  $X$  is path connected and  $x, y \in X$ . Then  $\pi_1(X, x)$  is isomorphic to  $\pi_1(X, y)$ .

*Proof.* Let  $\gamma$  be a path from  $x$  to  $y$ . Define

$$F_\gamma: \pi_1(X, y) \rightarrow \pi_1(X, x) \text{ by } F_\gamma(\alpha) = [\gamma] \cdot [\alpha] \cdot [\gamma^{-1}]$$

Note that  $\gamma$  is not a loop and the equivalence class  $[\gamma]$  is not an element of a group. Nevertheless, it is easy to see that  $F_\gamma$  is a well defined and maps the equivalence classes of loops based at  $y$  into those based at  $x$ . It is a group homomorphism since

$$F_\gamma([\alpha]) \cdot F_\gamma([\beta]) = [\gamma] \cdot [\alpha] \cdot [\gamma^{-1}] \cdot [\gamma] \cdot [\beta] \cdot [\gamma^{-1}] = [\gamma] \cdot [\alpha] \cdot [\beta] \cdot [\gamma^{-1}] = F_\gamma([\alpha] \cdot [\beta])$$

Finally it has an inverse namely  $F_{\gamma^{-1}}$ : that is  $F_{\gamma^{-1}} \circ F_\gamma$  and  $F_\gamma \circ F_{\gamma^{-1}}$  are both the identity.  $\square$

*Remark.* In one sense we can now talk about the fundamental group  $\pi_1(X)$  since it does not depend on the base point. But note this is a little sloppy: the group is not the same group but they are isomorphic. In other words if we want to know about a group property it does not matter which basepoint we use. BUT if we want to ask things like ‘is a particular loop in the group’ we need the actual group so we need a fixed basepoint.

*Remark.* Note that the isomorphism above may depend on the path from  $x$  to  $y$  used in the proof.

**Definition.** Suppose  $X$  and  $Y$  are topological spaces and  $f: X \rightarrow Y$  is continuous. The induced map  $f_*$  is the map  $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  defined by  $f_*([\alpha]) = [f \circ \alpha]$ .

**Theorem 51.** The map  $f_*$  is well defined and is a group homomorphism. Moreover,  $(g \circ f)_* = g_* \circ f_*$ .

### Summary of proof

1. If  $\alpha \simeq \alpha'$  and  $F$  is a homotopy relating these two paths then  $f \circ F$  is a homotopy relating  $f \circ \alpha$  and  $f \circ \alpha'$  (obviously,  $f \circ \alpha$  is a loop in  $Y$ ). Hence  $[f \circ \alpha] = [f \circ \alpha']$  which means that  $f_*$  is well defined.
2.  $f \circ (\alpha \cdot \beta) = (f \circ \alpha) \cdot (f \circ \beta)$  follows from the definition of the join operation (you are supposed to write down the relevant formulae).
3. Since  $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$  (by the definition of multiplication in  $\pi_1(X, x)$ ) we have  $f_*([\alpha] \cdot [\beta]) = f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)]$  (the last step is due to the definition of  $f_*$ ).
4. Next by 2, we have  $[f \circ (\alpha \cdot \beta)] = [(f \circ \alpha) \cdot (f \circ \beta)] = [f \circ \alpha] \cdot [f \circ \beta] = f_*([\alpha]) \cdot f_*([\beta])$ .
5. Finally  $(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)] = g_*([f \circ \alpha]) = g_*(f_*([\alpha])) = g_* \circ f_*([\alpha])$ .

**Corollary 52.** Suppose  $X$  and  $Y$  are homeomorphic. Then  $\pi_1(X)$  is isomorphic to  $\pi_1(Y)$ .

*Remark:* of course the converse is not true; that is two spaces can have the same fundamental group without being homeomorphic. Indeed, any two simply connected spaces have the same fundamental group: namely the trivial group on one element (all loops are homotopic) and e.g., the single point space and  $\mathbb{R}$  are not homeomorphic.



## The fundamental group of the circle $S^1$ is isomorphic to $\mathbb{Z}$ .

**Theorem 53.** The fundamental group  $\pi_1(S^1, 1)$  is isomorphic to  $\mathbb{Z}$ .

We think of the circle  $S^1$  as a subset of  $\mathbb{C}$  as normal. We proved in the class that if  $\alpha$  and  $\beta$  are paths in  $S^1$  given by  $\alpha(t) = e^{2\pi i n t}$  and  $\beta(t) = e^{2\pi i m t}$  respectively then  $\alpha \cdot \beta \simeq \gamma$ , where  $\gamma(t) = e^{2\pi i (n+m)t}$ . If you also assume that (a) every loop in  $S^1$  is homotopic to some  $\alpha$  (with  $n$  depending on this path) and (b) that  $\alpha \not\simeq \beta$  if  $n \neq m$ , then this proves Theorem 53.

### Proof of Theorem 53.

First the setup. We have  $\mathbb{R}$  and the circle  $S^1$  which we think of as a subset of  $\mathbb{C}$  as normal and we have the quotient map  $q: \mathbb{R} \rightarrow S^1$  given by  $q(t) = e^{2\pi i t}$ . Note that  $q(s) = q(t)$  if and only if  $s - t$  is an integer.

The usual definition of a path  $\alpha$  in  $S^1$  in fact means that  $\alpha(t) = x(t) + iy(t) \in S^1$ , where  $x(\cdot), y(\cdot)$  are continuous functions of  $t \in [0, 1]$  (with  $x(t)^2 + y(t)^2 = 1$ ).

Next, we shall make use (without proof) of the following statement from analysis:

**Lemma 54.** For a given path  $\alpha$  in  $S^1$  and a  $t_0 \in [0, 1]$  there is an  $\varepsilon > 0$  and a continuous function  $g: (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow \mathbb{R}$  such that  $e^{2\pi i g(t)} = \alpha(t)$  for all  $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ . Moreover, if for some  $\tilde{t} \in (t' - \varepsilon, t' + \varepsilon)$  and some  $\tilde{g}$  one has  $e^{2\pi i \tilde{g}} = \alpha(\tilde{t})$  then the function  $g$  satisfying  $g(\tilde{t}) = \tilde{g}$  is unique.

*Exercise.* Prove this lemma. (The proof was in fact explained in the class.)

**Definition** A path  $\hat{\alpha}$  in  $\mathbb{R}$  is said to be a lift of a path  $\alpha$  in  $S^1$  if  $q \circ \hat{\alpha} = \alpha$ .

Simply said, a path  $\hat{\alpha}$  in  $\mathbb{R}$  is a lift of  $\alpha$  in  $S^1$  if  $e^{2\pi i \hat{\alpha}(t)} = \alpha(t)$  for all  $t \in [0, 1]$ .

*Remark.* If  $\hat{\alpha}$  is a lift of  $\alpha$  then any function  $g: [0, 1] \rightarrow \mathbb{R}$  such that  $g(t) - \hat{\alpha}(t) = n(t)$ , where  $n(t)$  is an integer has the property  $q \circ g = \alpha$ . Note that  $g$  does not have to be continuous.

*Exercise.* Prove that  $g$  is a lift if and only if  $n(t) = \text{const}$  (does not depend on  $t$ ).

**Lemma 55** (Path Lifting Lemma for  $S^1$ ). Suppose  $\alpha$  is a loop in  $S^1$  based at 1. Then there exists a unique path  $\hat{\alpha}$  in  $\mathbb{R}$  with

(i)  $q \circ \hat{\alpha} = \alpha$ ,

(ii)  $\hat{\alpha}(0) = 0$ .

(Obviously,  $\hat{\alpha}$  is a lift of  $\alpha$ ).

*Proof.* Cover each point  $t_0 \in [0, 1]$  by an interval described in Lemma 54. These intervals form an open cover of  $[0, 1]$  which, since  $[0, 1]$  is a compact set, has a finite subcover, say

$$\{I_j \equiv (t_j - \varepsilon_j, t_j + \varepsilon_j) : j = 1, \dots, m\}.$$

We can always assume that  $t_1 < t_2 < \dots < t_m$  and that  $I_j \cap I_{j+2} = \emptyset$  for all  $j$ . (This cover is "minimal" in the sense that each of these intervals covers some points which do not belong to any other interval.) We can now construct the path  $\widehat{\alpha}$  as follows. By Lemma 54 there is a unique function  $g_1 : I_1 \rightarrow \mathbb{R}$  such that  $g_1(0) = 0$  and  $e^{2\pi i g_1(t)} = \alpha(t)$  for all  $t \in I_1$ ; set  $\widehat{\alpha}(t) = g_1(t)$  on  $I_1$ . Next, proceed by induction: if  $\widehat{\alpha}$  is defined on  $\cup_{i=1}^j I_i$  then first choose  $\tilde{t}_j \in I_j \cap I_{j+1}$ ; next by Lemma 54 there is  $g_{j+1} : I_{j+1} \rightarrow \mathbb{R}$  with  $e^{2\pi i g_{j+1}(t)} = \alpha(t)$  for all  $t \in I_{j+1}$  and  $g_{j+1}(\tilde{t}_j) = g_j(\tilde{t}_j)$ ; finally set  $\widehat{\alpha}(t) = g_{j+1}(t)$  on  $I_{j+1}$ . Note that so constructed  $\widehat{\alpha}$  is continuous and hence the existence of  $\widehat{\alpha}$  is proved.

Uniqueness. Suppose that  $\widehat{\alpha}'$  is another path with  $\widehat{\alpha}'(0) = 0$ . Then by Lemma 54 these two paths are the same on some interval containing 0. Let  $\bar{t} = \sup\{t : \widehat{\alpha}(t) = \widehat{\alpha}'(t)\}$ . If  $\bar{t} = 1$  then the Lemma is proved. If  $\bar{t} < 1$  then by Lemma 54 there is  $\varepsilon > 0$  such that  $\widehat{\alpha}(t) = \widehat{\alpha}'(t)$  when  $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$ . Hence  $\widehat{\alpha}(t) = \widehat{\alpha}'(t)$  for all  $t \in [0, 1]$ .  $\square$

**Lemma 56** (Homotopy lifting lemma for  $S^1$ ). Suppose  $\alpha$  and  $\beta$  are loops in  $P_1(S^1, 1)$  with  $\alpha \simeq \beta$ . Then  $\widehat{\alpha} \simeq \widehat{\beta}$ .

This proof of Lemma 56 is NOT EXAMINABLE. The idear of the proof was briefly explained in the class; it is similar to that of Lemma 55.

**We can now finish the proof of Theorem 53.**

### Summary of proof

1. Define  $J : \pi_1(S^1, 1) \rightarrow \mathbb{Z}$  by  $J([\alpha]) = \widehat{\alpha}(1)$ .  
If  $\alpha \simeq \alpha'$  then by homotopy lifting lemma  $\widehat{\alpha} \simeq \widehat{\alpha}'$  so  $\widehat{\alpha}(1) \simeq \widehat{\alpha}'(1)$  so  $J$  is well defined.
2. If  $J([\alpha]) = J([\beta])$  then  $\widehat{\alpha}(1) = \widehat{\beta}(1)$ . Since we also have  $\widehat{\alpha}(0) = \widehat{\beta}(0)$  we have  $\widehat{\alpha} \simeq \widehat{\beta}$ . Hence  $\alpha = q \circ \widehat{\alpha} \simeq q \circ \widehat{\beta} = \beta$ ; i.e.,  $J$  is injective .
3. Suppose  $m \in \mathbb{Z}$ . Then the loop  $\alpha_m$  given by  $\alpha_m(t) = e^{2m\pi it}$  lifts to the path  $mt$  so  $J([\alpha_m]) = m$ ; hence  $J$  is surjective.
4. We proved before that  $\alpha_m \cdot \alpha_n \simeq \alpha_{m+n}$ . By the lifting lemma, for any  $\alpha$  and  $\beta$  there are  $m$  and  $n$  such that  $\alpha \simeq \alpha_m$  and  $\beta \simeq \alpha_n$ . We then have that

$$J([\alpha] \cdot [\beta]) = J([\alpha_m] \cdot [\alpha_n]) = J([\alpha_m \cdot \alpha_n]) = J([\alpha_{m+n}]) = m + n = J([\alpha]) + J([\beta])$$

which means that  $J$  is a group homomorphism and the proof is complete.

## Applications of the Fundamental Group

### Brouwer fixed point theorem

**Definition.** Suppose  $X$  is a topological space and  $Y$  is a subspace of  $X$ . A retraction from  $X$  to  $Y$  is a continuous map from  $X$  to  $Y$  with  $f(y) = y$  for all  $y \in Y$ .

*Examples.* There is an obvious retraction of  $\mathbb{R}^2$  to the  $x$ -axis (i.e., to the set  $\{(x, 0) \in \mathbb{R}^2\}$ ) by mapping  $(x, y)$  to  $(x, 0)$ .

*Examples.* In the above situation the map  $(x, y) \mapsto (-x, 0)$  is not a retraction. It is continuous and maps  $X$  into  $Y$  but the  $x$ -axis is not preserved by the map: e.g.,  $(1, 0)$  maps to  $(-1, 0)$ .

*Examples.* Let  $X = [0, 1]$  and  $Y = \{0, 1\}$ . There is no retraction from  $X$  to  $Y$ . Indeed,  $X$  is path connected and  $Y$  is not so there is no continuous surjection  $X$  to  $Y$ . (Another, more direct, explanation was given in the lecture.)

**Theorem 57** (Brouwer fixed point theorem.). Suppose that  $f: B^2 \rightarrow B^2$  is continuous. Then  $f$  has a fixed point.

*Proof.* We first prove that  $\pi_1(B^2, 0)$  is trivial. Suppose that  $\alpha \in P_1(B^2, 0)$ . Then the map  $F: [0, 1] \times [0, 1] \rightarrow B^2$  given by  $F(t, s) = s\alpha(t)$  is obviously continuous and  $F(t, 0) = 0$ ,  $F(t, 1) = \alpha(t)$ ,  $F(0, s) = 0$  and  $F(1, s) = 0$  so  $F$  is a homotopy relative  $\{0, 1\}$  between  $\alpha$  and the constant zero path. Hence any two loops with base at 0 are homotopic relative  $\{0, 1\}$ .

Next, we prove the following lemma.

**Lemma 58.** There is no continuous retraction from  $B^2$  to  $S^1$ .

*Proof.* Suppose there is. Call this map  $g$ . We know that  $\pi_1(S^1) = \mathbb{Z}$  and that  $\pi_1$  is independent of the base point taken. We fix the base point to be the point  $1 \in \mathbb{C}$ . Then  $g$  induces a group homomorphism  $g_*: \pi_1(B^2, 1) \rightarrow \pi_1(S^1, 1)$ .

Let  $\alpha$  be a path in  $S^1$  corresponding to any non-identity group element in  $\mathbb{Z}$ . View  $\alpha$  as a path in  $B^2$ . Then  $g_*(\alpha) = g \circ \alpha = \alpha$  since  $g$  fixes  $S^1$ . In particular the image of the map  $g_*$  contains the non-identity element  $\alpha$ . This contradicts the fact that  $\pi_1(B^2)$  is trivial  $\pi_1(B^2) = \{e\}$ .  $\square$

We are now in a position to finish the proof of the theorem. Suppose that  $f: B^2 \rightarrow B^2$  has no fixed point. Define  $g: B^2 \rightarrow S^1$  as follows. Take the straight line from  $f(x)$  to  $x$  and extend (in this direction) to the circle  $S^1$ . This is well defined since  $f(x) \neq x$  and is continuous. Moreover, it fixes every point of  $S^1$ , i.e., it is a retraction which contradicts the lemma.  $\square$

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*Remark.* In fact this is true in higher dimensions: but the techniques in this course cannot prove it. Our techniques use loops which are essentially one dimensional and homotopies which are sort of two dimensional. So our techniques mostly prove results about ‘2-dimensional’ objects.

## Borsuk-Ulam Theorem

**Definition.** Suppose that  $f: S^n \rightarrow S^m$ . We say that  $f$  preserves antipodal points if  $f(-x) = -f(x)$ . That is opposite points in  $S^n$  map to opposite points in  $S^m$ .

*Examples.* Suppose that  $f: S^1 \rightarrow S^1$  is a continuous map preserving antipodal points. Can we think of any examples? Yes: the identity map. Consider  $S^1$  as a subset of  $\mathbb{C}$  as normal; then the map  $z \mapsto z^2$  does not preserve antipodal points. However the map  $z \mapsto z^3$  does preserve antipodal points.

**Lemma 59.** Suppose that  $f: S^1 \rightarrow S^1$  is a continuous map preserving antipodal points. Then the map  $f_*: \pi_1(S^1, 1) \rightarrow \pi_1(S^1, f(1))$  is not trivial.

*Remark.* We could equally well say that  $f_*: \pi_1(S^1) \rightarrow \pi_1(S^1)$  is not trivial. These are equivalent statements since groups corresponding to different base points are isomorphic.

*Proof.* Consider the loop  $\alpha(t) = e^{2\pi it}$ . Without loss of generality we suppose that  $f \circ \alpha(0) = 1$  and hence  $f \circ \alpha(1/2) = -1$ .

Consider the lift  $g$  of the loop  $f \circ \alpha$ . Then  $g(0) = 0$  by definition and  $g(1/2) = m + 1/2$  for some  $m \in \mathbb{Z}$ . Moreover the map  $g'$  defined by

$$g'(t) = \begin{cases} g(t) & \text{for } t \leq 1/2 \\ g(t - 1/2) + g(1/2) & \text{for } t \geq 1/2 \end{cases}$$

is also a lift. Indeed,  $g'$  is continuous by the gluing lemma. Also it is trivial that  $q \circ g'(t) = q \circ g(t)$  for  $t \leq 1/2$  (remember that  $q(x) = e^{2\pi ix}$ ). Next, for  $t \geq 1/2$  we have

$$\begin{aligned} q \circ g'(t) &= q(g'(t)) \\ &= q(g(t - 1/2) + g(1/2)) \\ &= q(g(t - 1/2) + m + 1/2) \\ &= -q(g(t - 1/2)) = -f \circ \alpha(t - 1/2) \\ &= f \circ \alpha(t). \end{aligned}$$

(since  $\alpha(t)$  and  $\alpha(t - 1/2)$  are antipodal). Hence  $g' = g$ . In particular  $g(1) = 2g(1/2) = 2(m + 1/2) = 2m + 1$  which is not zero; that is it is not the trivial loop.  $\square$

*Remark.* In fact we showed that  $f_*$  maps the loop  $e^{2\pi it}$  into with a lift homotopic to  $(2m + 1)t$ .

**Theorem 60** (Borsuk-Ulam Theorem). Suppose that  $f: S^2 \rightarrow \mathbb{R}^2$  is continuous. Then there exists  $x$  such that  $f(x) = f(-x)$ .

*Proof.* Suppose that for all  $x$  with  $f(x) \neq f(-x)$ . Then define  $h: S^2 \rightarrow S^1$  by

$$h(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

which is defined and continuous. Moreover it preserves antipodal points.

Now the path around the equator ( $\sim S^1$ ) gets mapped to a path around  $S^1$ . This is a non-trivial path. But  $S^2$  is simply connected: i.e., has trivial fundamental group. Contradiction.  $\square$

*Remark.* We may plausibly imagine that the temperature and pressure on the surface of the earth vary continuously. Hence if we map each point on the surface of the earth to the pair (temperature, pressure) we have a continuous function  $S^2 \rightarrow \mathbb{R}^2$ . Hence the Borsuk-Ulam theorem says that there are always two antipodal points with the same temperature and pressure!