Topology
Exercise sheet 7

1. Prove that the relation of homotopic relative $\{0, 1\}$ (that is $\simeq$) is an equivalence relation on paths starting at one fixed point and finishing at another fixed point. There are three things to prove:

   (a) reflexivity (easy);

   (b) symmetry (reasonable);

   (c) transitivity (harder).

2. Let α be the path in $\mathbb{R}$ given by

$$\alpha(t) = \begin{cases} 2t & t \leq 1/2 \\ 2 - 2t & t \geq 1/2. \end{cases}$$

   (a) Prove α is continuous.

   (b) Let β be the path in $\mathbb{R}$ given by $\beta(t) = 0$ for all t. Prove that $\alpha \simeq \beta$.

3. Let X be the space formed by taking $[0, 1]$ and identifying 0 and 1 (and no other points). Prove that X is homeomorphic to S^1.

4. Prove that D^2 and $\mathbb{R}^2$ are homeomorphic.

5. Suppose X is a compact subset of $\mathbb{R}$. We define the topological cone on X as follows: take $X \times [0, 1]$ and identify all points $(x, 1)$ together. We define the geometric cone as a subset of $\mathbb{R}^2$ by taking a copy of X in the x-axis, and then joining each point by a straight line to the point $(0, 1)$. [Draw a picture.]

   Prove that the geometric cone and the topological cone are homeomorphic.