Topology

Exercise sheet 4

There are lots (and lots) of questions: you do not need to do them all but you should think whether you could do them. Ideally, if I asked you in the tutorial how to do a question you would be able to answer it. To solve problems 6-10 you need to know that a set $A \subset X$ is called compact if every cover of A by open sets has a finite sub-cover.

1. Suppose that X and Y are topological spaces and that f is a function $f\colon X \to Y$. Prove that f is continuous if and only if the pre-image $f^{-1}(V)$ of every closed set V in Y is closed (in X).

2. Which of the following spaces are path connected.
   (a) $\{(x,y) \in \mathbb{R}^2 \ : \ x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$
   (b) $\{(x,y) \in \mathbb{R}^2 \ : \ x \in \mathbb{Q} \text{ and } y \in \mathbb{Q}\}$
   (c) a set X with the discrete topology
   (d) a set X with the indiscrete topology
   (e) (harder) $\{(x, \sin 1/x) \ : \ x > 0\} \cup \{(0,y) \ : \ y \in \mathbb{R}\}$

3. Give examples of subsets of $\mathbb{R}$ with one, two and infinitely many path-components. Is there an example with uncountably many path-components?

4. Prove that the number of path-components is a topological property. Deduce that a space shaped like a 'T' is not homeomorphic to $[0,1]$. (Formally you could view the 'T' as $\{(x,0)\colon x \in [-1,1]\} \cup \{(0,y) : y \in [-1,0]\}$).

5. Prove the glueing lemma but for the situation that A and B are both *open* in X (rather than closed). Give an example to show that the glueing lemma is not true if we do not impose some condition on A and B.

6. Suppose that A is a subset of $\mathbb{R}$ which is not closed. Prove that A is not compact. Deduce that if X is compact and $f\colon X \to \mathbb{R}$ is continuous then f attains its bounds: that is there is an $x \in X$ such that $f(y) \leq f(x)$ for all $y \in X$.

7. Suppose that X is a topological space and that A is a compact subset of X. We can also view A as a topological space in its own right (with the subspace topology). Prove that A is compact in this topology. [This shows that compactness is an intrinsic property of A: it does not matter what space it is in.]

8. Suppose that X is compact and that A is a closed subset of X. Prove that A is compact.

9. Suppose that X is a compact topological space and that F_i for $i \in I$ is a collection of closed sets with the property that for any finite collection $F_1, F_2, F_3, \ldots, F_n$ of these closed sets we have $\bigcap_{i=1}^n F_i \neq \emptyset$. Prove that $\bigcap_{i \in I} F_i \neq \emptyset$.

10. When is a set X with the discrete topology compact? When is X with the indiscrete topology compact? What about the co-finite and co-countable topologies?

11. Prove that the closed interval $[0,1]$ is not homeomorphic to $\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$ (the unit circle in $\mathbb{R}^2$).