Topology
Exercise sheet 1

1. Prove that the following sets are countable.
   (a) Any infinite subset of a countable set.
   (b) A direct product of two countable sets.
   (c) The set of all rational numbers.
   (d) A countable union of countable sets.
   (e) The set of all algebraic numbers.
2. Prove that $\mathbb{R}$ has the same cardinality as $\mathbb{R}^2$.
3. Suppose X is an infinite set. A set U is open if either $U = \emptyset$ or $X \setminus U$ contains finitely many points. Check that if $\mathcal{T}$ is a collection of such sets then it is a topology. We call this the co-finite topology.
4. Consider a co-finite topology on a finite set. What would the topology be? (i.e., what are the open sets)
5. List all possible topologies on the sets $\{a, b\}$ and $\{a, b, c\}$.
6. Find two topologies $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{R}$ such that there is a set $\mathcal{F}_1$-open set that is not $\mathcal{F}_2$-open and a set V that is $\mathcal{F}_2$ open but not $\mathcal{F}_1$ open.
7. Define the co-countable topology as follows. Suppose that X is an infinite set. A set U is open if either $U = \emptyset$ or $X \setminus U$ is countable. Prove that this is a topology.