MTH5121 Probability Models

Test 11th November 2011

The duration of this test is 40 minutes. Write your name and student number in the spaces below.

Electronic calculators are not permitted in this exam. The unauthorized use of calculator constitutes examination offence.

Answer all questions. Write all answers in the spaces provided. If you run out of space for an answer, continue on the back of the page.

Name: ______________________________

Student Number: ______________________________

1. X and Y are independent random variables with the same probability mass function, namely $P(X = 0) = 0.3$, $P(X = 2) = 0.7$

(a) $\boxed{\text{4 marks}}$ Write down the probability generating functions $G_X(t)$ and $G_Y(t)$.

(b) $\boxed{\text{4 marks}}$ Obtain the probability generating function of $Z = X + Y$.

(c) $\boxed{\text{6 marks}}$ Obtain the probability mass function for Z.

(d) $\boxed{\text{4 marks}}$ Write down the formulae for $E(X)$ and $Var(X)$ in terms of $G_X(t)$.

(e) $\boxed{\text{6 marks}}$ Using the definition $G_X(t) = E(t^X)$, prove these formulae.

(f) $\boxed{\text{6 marks}}$ Write down the probability generating function for a random variable $X \sim Geometric(p)$ and use it to find $E(X)$.

2. $\boxed{\text{24 marks}}$ A gambler has a pot of £12. He plays a series of games. At each game he has probability p of winning and probability $(1-p)$ of losing. He pays £6 every time he loses a game and he is paid £6 every time he wins the game.

(a) Calculate the probability that he wins an infinite amount of money if $p = 0.52$.

(b) Calculate the probability that he loses all his money.

Remarks 1. Obviously, to win £∞ one needs infinite time which means that the gambler is allowed to play for as long as he wishes.

2. You are supposed to use, without proof, the following formula for the probability of a simple random walk starting from n to reach N before M:

$$r_n(M,N) = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}, \quad \text{where } M \le n \le N.$$

3. Conditional Expectations.

(a) $\boxed{\text{4 marks}}$ State the total probability formula for expectations.

(b) $\boxed{\text{6 marks}}$ Define the random variable which is the conditional expectation $E(X|Y)$, where X and Y are two (discrete) random variables.

(c) $\boxed{\text{14 marks}}$ Prove the following Theorem:
Theorem $E(X) = E[E(X|Y)]$.

4. $\boxed{\text{12 marks}}$ Suppose that a random number N of Bernoulli trials is carried out and that $P(N=k) = pq^k$, $k = 0, 1, 2, \ldots$ (as usual, $p+q=1$). The probability of success in each trial is 0.5. Denote by Y the total number of successes.

   It is easy to show that $G_N(t) = \frac{p}{1-qt}$ (you are supposed to use this fact without proving it).

   Find $G_Y(t)$ and conclude that $P(Y=k) = \bar{p}\bar{q}^k$. State the value of $\bar{p}$.

5. $\boxed{\text{10 marks}}$ Consider a Branching Process whose generating distribution is given by $P(X=0) = 0.3$, $P(X=2) = 0.7$ (the same random variable as in problem 1). Suppose that $Y_0 = 1$.

   (a) Find the probability generating function for Y_2.

   (b) Hence find the probability mass function for Y_2 and in particular the probability that by time 2 the process will be extinct.