MTH5121 Probability Models

Test   11th November 2011

The duration of this test is 40 minutes. Write your name and student number in the spaces below.

Electronic calculators are not permitted in this exam. The unauthorized use of calculator constitutes examination offence.

Answer all questions. Write all answers in the spaces provided. If you run out of space for an answer, continue on the back of the page.

Name: ______________________________

Student Number: ______________________________

1. X and Y are independent random variables with the same probability mass function, namely $P(X=0)=0.3$, $P(X=2)=0.7$

(a) $\boxed{\text{4 marks}}$ Write down the probability generating functions $G_X(t)$ and $G_Y(t)$.

Solution By the definition of a p.g.f. we have

$$G_X(t) = G_Y(t) = 0.3 + 0.7t^2.$$

(b) $\boxed{\text{4 marks}}$ Obtain the probability generating function of $Z = X + Y$.

Solution Since X an Y are independent r. v.s. we have

$$G_Z(t) = G_{X+Y}(t) = G_X(t)G_Y(t) = (0.3 + 0.7t^2)^2.$$

(c) $\boxed{\text{6 marks}}$ Obtain the probability mass function for Z.

Solution By the definition of the p.g.f. of Z

$$G_Z(t) = \sum_{j=0}^{\infty} P(Z=j)t^j \equiv P(Z=0) + P(Z=1)t + P(Z=2)t^2 + \ldots$$

On the other hand we know that

$$G_Z(t) = 0.09 + 0.42t^2 + 0.49t^4.$$

Hence

$$P(Z=0) = 0.09, \quad P(Z=2) = 0.42, \quad P(Z=4) = 0.49$$

and $P(Z=j) = 0$ for all other values of j.

(d) $\boxed{\text{4 marks}}$ Write down the formulae for $E(X)$ and $Var(X)$ in terms of $G_X(t)$.
Solution $E(X) = G'_X(1),\ Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$.

(e) $\boxed{\text{6 marks}}$ Using the definition $G_X(t) = E(t^X)$, prove these formulae.
Proof From $G_X(t) = E[t^X]$ we have

$$G'_X(t) = \frac{d}{dt}E[t^X] = E[(t^X)'] = E[Xt^{X-1}]$$

and therefore $G'_X(1) = E[X]$ which proves the first relation. Similarly,

$$G''_X(t) = \frac{d}{dt}E[Xt^{X-1}] = E[(Xt^{X-1})'] = E[X(X-1)t^{X-1}]$$

and therefore $G''_X(1) = E[X(X-1)]$. Note that $E[X(X-1)] = E[X^2] - E[X]$ and hence $E[X^2] = E[X(X-1)] + E[X] = G''_X(1) + G'_X(1)$. By the definition of the variance and due to these relations we obtain

$$Var(X) = E[X^2] - (E[X])^2 = G''_X(1) + G'_X(1) - [G'_X(1)]^2. \quad \square$$

(f) $\boxed{\text{6 marks}}$ Write down the probability generating function for a random variable $X \sim Geometric(p)$ and use it to find $E(X)$.
Solution

$$G_X(t) = \sum_{k=1}^{\infty}(pt)(qt)^{k-1} = \frac{pt}{1-qt}$$

and therefore $G'_X(t) = \frac{p(1-qt)-pt(-q)}{(1-qt)^2} = \frac{p}{(1-qt)^2}$. Hence, according to the above formula for $E(X)$, we have

$$E(X) = G'_X(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

2. $\boxed{\text{24 marks}}$ A gambler has a pot of £12. He plays a series of games. At each game he has probability p of winning and probability $(1-p)$ of losing. He pays £6 every time he loses a game and he is paid £6 every time he wins the game.

(a) Calculate the probability that he wins an infinite amount of money if $p = 0.52$.

(b) Calculate the probability that he loses all his money.

Remarks 1. Obviously, to win £∞ one needs infinite time which means that the gambler is allowed to play for as long as he wishes.

2. You are supposed to use, without proof, the following formula for the probability of a simple random walk starting from n to reach N before M:

$$r_n(M,N) = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}, \quad \text{where } M \le n \le N.$$

Solution (a) By definition, the probability of winning an infinite amount of money is equal to the $\lim_{N\to\infty} r_n(M,N)$. To find this limit, note first that in our case

$$q = 1 - p = 0.48 \text{ and } \frac{q}{p} = \frac{12}{13} < 1.$$

Also in our case $n = \frac{12}{6} = 2$, $M = \frac{0}{6} = 0$. Note also that we should write $N/6$ instead of N in the above formula. The answer is thus given by

$$\lim_{N\to\infty} r_n(M,N) = \lim_{N\to\infty} \frac{\left(\frac{12}{13}\right)^2 - \left(\frac{12}{13}\right)^0}{\left(\frac{12}{13}\right)^{N/6} - \left(\frac{q}{p}\right)^0} = \lim_{N\to\infty} \frac{\left(\frac{12}{13}\right)^2 - 1}{\left(\frac{12}{13}\right)^{N/6} - 1} = \frac{\left(\frac{12}{13}\right)^2 - 1}{0-1} = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169}.$$

We use the fact that $\lim_{K\to\infty} \theta^K = 0$ if $|\theta| < 1$ (obviously, $K = N/6 \to \infty$ as $N \to \infty$).

(b) The probability

$$P\{\text{ he eventually looses all his money }\} = 1 - P\{\text{ he wins £}\infty\} = \left(\frac{12}{13}\right)^2.$$

Remark. We use here the fact that the probability that none of these two events will take place is 0. The rigorous proof of this fact follows from a direct calculation similar to the one in (a):

$$P\{\text{ he eventually looses all his money }\} = \lim_{N\to\infty} l_n(M,N) = \lim_{N\to\infty}\left(\frac{\left(\frac{12}{13}\right)^{N/6} - \left(\frac{12}{13}\right)^2}{\left(\frac{12}{13}\right)^{N/6} - 1}\right) = \left(\frac{12}{13}\right)^2$$

which gives one more solution to this question.

3. Conditional Expectations.

(a) $\boxed{\text{4 marks}}$ State the total probability formula for expectations.

Solution If X is a random variable and events $B_1, \ldots, B_n$ partition S then

$$E[X] = \sum_{j=1}^{n} E[X|B_j]P(B_j).$$

(b) $\boxed{\text{6 marks}}$ Define the random variable which is the conditional expectation $E(X|Y)$, where X and Y are two (discrete) random variables.

Definition. Consider the function $H(y) = E[X|Y = y]$. The random variable $H(Y)$ is called the conditional expectation of X conditioned on Y and is denoted $E[X|Y]$. Equivalently, one could say that $E[X|Y]$ is a function of Y which takes the value $E[X|Y = y]$ when $Y = y$.

(c) $\boxed{\text{14 marks}}$ Prove the following Theorem:

Theorem $E(X) = E[E(X|Y)]$.

Proof Since $E[E[X|Y]]$ is a function of Y (namely, $H(Y)$) we can use the usual formula

$$E[E[X|Y]] = E[H(Y)] = \sum_{y_j} H(y_j)P(Y = y_j) = \sum_{y_j} E[X|Y = y_j]P(Y = y_j).$$

Since the evens $(Y = y_j)$ form a partition, we have that

$$\sum_{y_j} E[X|Y = y_j]P(Y = y_j) = E(X)$$

due to the Total Probability Law for expectations. $\square$

4. $\boxed{\text{12 marks}}$ Suppose that a random number N of Bernoulli trials is carried out and that $P(N=k) = pq^k$, $k = 0,1,2,...$ (as usual, $p+q=1$). The probability of success in each trial is 0.5. Denote by Y the total number of successes.

   It is easy to show that $G_N(t) = \frac{p}{1-qt}$ (you are supposed to use this fact without proving it).

   Find $G_Y(t)$ and conclude that $P(Y=k) = \bar{p}\bar{q}^k$. State the value of $\bar{p}$.

   Solution Let X_j be the number of successes in j^{th} trial. Then

$$Y = X_1 + X_2 + ... + X_N.$$

   In other words, Y is a random sum. We proved that the p.g.f. for random sums is given by

$$G_Y(t) = G_N(G_X(t)),$$

   where X has the same distribution as X_j (see Notes 3). Obviously, $G_X(t) = 0.5 + 0.5t$. Hence

$$G_Y(t) = G_N(G_X(t)) = \frac{p}{1-qG_X(t)} = \frac{p}{1-q(0.5+0.5t)},$$

   which gives the answer to the first part of the question. Next,

$$G_Y(t) = \frac{p}{1-q(0.5+0.5t)} = \frac{\frac{p}{1-0.5q}}{1-\frac{0.5q}{1-0.5q}t)} = \frac{\bar{p}}{1-\bar{q}t},$$

   where $\bar{p} = \frac{p}{1-0.5q}$ (and $\bar{q} = \frac{0.5q}{1-0.5q}$; check that $\bar{p}+\bar{q} = 1$). This in turn implies that $P(Y=k) = \bar{p}\bar{q}^k$.

5. $\boxed{\text{10 marks}}$ Consider a Branching Process whose generating distribution is given by $P(X=0) = 0.3$, $P(X=2) = 0.7$ (the same random variable as in problem 1). Suppose that $Y_0 = 1$.

   (a) Find the probability generating function for Y_2.

   (b) Hence find the probability mass function for Y_2 and in particular the probability that by time 2 the process will be extinct.

   Solution (a) The generating r. v. of this BP has p.g.f. $G(t) = 0.3+0.7t^2$. As usual, $G_1(t) \equiv G_{Y_1}(t)G(t) = 0.3+0.7t^2$ and, as we know, $G_{n+1}(t) = G(G_n(t))$. Hence the p.g.f. for Y_2 is given by

$$G_2(t) = G(G(t)) = 0.3+0.7G(t)^2 = 0.3+0.7(0.3+0.7t^2)^2 = 0.363+0.294t^2+0.343t^4.$$

   (b) The above formula for $G_2(t)$ implies that

$$P(Y_2=0) = 0.363, \quad P(Y_2=2) = 0.294, \quad P(Y_2=4) = 0.343.$$

   In particular, this implies that the probability that by time 2 the process will be extinct is 0.363.