MTH5121 Probability Models

Test — 12th November 2010

The duration of this test is 40 minutes. Write your name and student number in the spaces below.

Electronic calculators may be used. The calculator must not be preprogrammed prior to the examination. Enter the name and type here ____________________

Answer all questions. Write all answers in the spaces provided. If you run out of space for an answer, continue on the back of the page.

Name: ____________________

Student Number: ____________________

1. X and Y are independent random variables with $X \sim Binomial\left(2, \frac{1}{3}\right)$ and $Y \sim Bernoulli\left(\frac{1}{2}\right)$.

(a) $\boxed{\text{6marks}}$ Write down the probability generating functions $G_X(t)$ and $G_Y(t)$.

Solution $G_X(t) = (pt+q)^n = (\frac{2}{3} + \frac{1}{3}t)^2$, $G_Y(t) = q + pt = \frac{1}{2} + \frac{1}{2}t$.

(b) $\boxed{\text{4marks}}$ Obtain the probability generating function of $Z = X + Y$.

Solution X and Y are independent and therefore $G_Z(t) \equiv G_{X+Y}(t) = G_X(t)G_Y(t) = (\frac{2}{3} + \frac{1}{3}t)^2(\frac{1}{2} + \frac{1}{2}t) = \frac{2}{9} + \frac{4}{9}t + \frac{5}{18}t^2 + \frac{1}{18}t^3$.

(c) $\boxed{\text{8marks}}$ Obtain the probability mass function for Z.

Solution $G_Z(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$, where $p_k = P\{Z = k\}$. Hence $P\{Z = 0\} = \frac{2}{9}$, $P\{Z = 1\} = \frac{4}{9}$, $P\{Z = 2\} = \frac{5}{18}$, $P\{Z = 3\} = \frac{1}{18}$.

(d) $\boxed{\text{10marks}}$ Let X and Y be independent random variables each with Poisson distribution, with parameters λ and μ respectively. Show that $Z = X + Y$ has Poisson distribution and state its parameter.

Solution Since $X \sim Poisson(\lambda)$ we have that $G_X(t) = e^{\lambda(t-1)}$. Similarly, $G_Y(t) = e^{\mu(t-1)}$. X and Y are independent and therefore $G_Z(t) \equiv G_{X+Y}(t) = G_X(t)G_Y(t) = e^{\lambda(t-1)}e^{\mu(t-1)} = e^{\lambda(t-1)+\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$. Hence $Z \sim Poisson(\lambda + \mu)$ (we use the fact that p.d.f. of Z is uniquely defined by the p.g.f. of Z.)

2. $\boxed{\text{30marks}}$ A gambler has a pot of £50. He plays a series of games. At each game he has probability p of winning and probability $(1-p)$ of losing. He pays £5 every time he loses a game and he is paid £5 every time he wins the game.

(a) Calculate the probability that he loses all his money before reaching £100 in the following cases: (i) $p = \frac{2}{5}$; (ii) $p = \frac{1}{2}$.

Solution (i) Since $p \neq q$ we can use the formula $l_n = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^n}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}$. In our case $q = 1 - p = \frac{3}{5}$ and hence $\frac{q}{p} = \frac{3}{2}$. $N = 100/5 = 20$, $n = 560/5 = 10$, $M = 0/5 = 0$. Thus

$$l_n = \frac{\left(\frac{3}{2}\right)^{20} - \left(\frac{3}{2}\right)^{10}}{\left(\frac{3}{2}\right)^{20} - 1} = \frac{\left(\frac{3}{2}\right)^{10}\left[\left(\frac{3}{2}\right)^{10} - 1\right]}{\left(\frac{3}{2}\right)^{20} - 1} = \frac{\left(\frac{3}{2}\right)^{10}}{\left(\frac{3}{2}\right)^{10} + 1}.$$

(ii)$p = q = 0.5$ Hence $l_n = \frac{N-n}{N-M} = \frac{20-10}{20-0} = 0.5$.

(b) Suppose that $p = 0.55$ and that the gambler decides that he will play until he loses all his money. What is the probability that he never looses his money (in other words, that he wins an infinite sum of money)?

Solution This time $\frac{q}{p} = \frac{0.45}{0.55} = \frac{9}{11} < 1$. Hence $\lim_{N\to\infty}\left(\frac{q}{p}\right)^N = 0$. Therefore

$$p_n = \lim_{N\to+\infty} p_n(M,N) = \lim_{N\to+\infty} \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{0 - \left(\frac{q}{p}\right)^M} = 1 - \left(\frac{q}{p}\right)^{n-M}$$

which in our case gives

$$p_{10} = 1 - \left(\frac{9}{11}\right)^{10}.$$

3. (a) $\boxed{\text{4marks}}$ State the total probability formula for expectations.

Solution If $B_1, \ldots, B_n$ partition S then

$$E[X] = \sum_{j=1}^{n} E[X|B_j]P(B_j).$$

(b) $\boxed{\text{18marks}}$ Let $M < n < N$ be integer numbers. A random walk starting from n jumps to the right with probability p and to the left with probability $q = 1 - p$. The length of each jump is 1. Let $E_n = E(T_n)$, where T_n is the time at which the random walk reaches M or N for the first time. Prove that

$$E_n = pE_{n+1} + qE_{n-1} + 1.$$

Solution Denote by B_1 and B_2 the events 'the walk's first step is to the right' and 'the walk's first step is to the left'. These events form a partition and by the total probability formula for expectations we have

$$E[T_n] = E[T_n|B_1]P(B_1) + E[T_n|B_2]P(B_2)$$

If the first step is to the right then 1 unit of time is spent for this step and the the walk still has to reach M or N starting from $n+1$. So $T_n|B_1 = 1 + T_{n+1}$, where T_{n+1} measures the time the walk spends for reaching M or N starting from $n+1$. Therefore $E[T_n|B_1] = 1 + E[T_{n+1}]$. Similarly $E[T_n|B_2] = 1 + E[T_{n-1}]$. Hence

$$E_n = p(1 + E_{n+1}) + q(1 + E_{n-1}) = pE_{k+1} + qE_{k-1} + 1$$

(since $p + q = 1$). $\square$

4. $\boxed{\text{12marks}}$ The number of customers arriving at a shop during one day is a r.v. $N \sim Poisson(200)$. The customers act independently of each other spending a random amount of X_j pounds in the shop. Given that $E(X_j) = 10$, $Var(X_j) = 20$, find the average value and the variance of the daily cash flow S in this shop.

Solution We deal here with a random sum $S = \sum_{j=1}^{N} X_j$, where S is the daily cash flow. Then $E[S] = E[X_j] \times E[N] = 20 \times 200 = 2000$. Next

$$Var(S) = Var(X_j)E[N] + E[X_j]^2 Var(N) = 20 \times 200 + 10^2 \times 200 = 24000.$$

(We use the fact that $E[N] = 100$ and $Var[N] = 100$.)

5. $\boxed{\text{8marks}}$ State the definition of a Branching Process.

Definition *Let X be an integer-valued non-negative r.v. with p.m.f. $p_k = P\{X = k\}$, $k = 0, 1, 2, \ldots$. We say that a sequence of random variable Y_n, $n = 0, 1, 2, \ldots$, is a BP if*
1. $Y_0 = 1$
2. $Y_{n+1} = X_1^{(n)} + X_2^{(n)} + \ldots + X_{Y_n}^{(n)}$,
where all r.v.'s $X_j^{(n)}$ have the same distribution as X and are independent of each other.
We say that the distribution of X is the generating distribution of the BP.

 End of test.