Probability Models - MINIMAL REQUIREMENTS FOR TEST.

Knowing the material listed below should be sufficient for a pass in the test. "Knowing" means
(a) to be able to state the result (definition, formula, Theorem, etc),
(b) to be able to use the result while solving relevant problems,
(c) to be able to prove the result.

1. P.g.f.:

(a) Know the definition of p.g.f., know p.g.f. for standard discrete distributions and be able to derive these formulae.

(b) Know how to find $P(X = k)$ by differentiating the p.g.f.

(c) Know how to find $E[X]$ and $Var(X)$ by differentiating the p.g.f. and be able to prove the corresponding formulae.

(d) Know that the p.g.f. of the sum of independent r.v.'s is the product of the individual p.g.f.'s.

2. Know the Theorem of Total Probability and be able to do simple examples.

Be able to prove the following

Theorem *The probabilities r_n, $M \leq n \leq M$ satisfy the following system of equations:*

$$r_n = pr_{n+1} + qr_{n-1}, \text{ if } M < n < N \quad (1)$$

$$r_M = 0, \quad r_N = 1 \quad (2)$$

Know the random walk/gambler's ruin result: $r_k \equiv r_k(M,N) = \frac{(\frac{q}{p})^k - (\frac{q}{p})^M}{(\frac{q}{p})^N - (\frac{q}{p})^M}$ if $p \neq q$ and $r_k = \frac{k-M}{N-M}$ if $p = q = \frac{1}{2}$. Be able to use these results in simple examples.

Know how to compute $\lim_{N\to\infty} r_k(M,N)$ and be able to prove the corresponding result. Be able to use this results in concrete examples (see CW3). Be able to state the relevant results for $M \to -\infty$.

3. Know the Theorem of Total Probability for Expectations. You are required to be able to derive the equations satisfied by $E(T_n)$, where T_n is the duration of the walk (but NOT their solutions).

4. Know results on conditional distributions and conditional expectations and be able to use them in simple examples.

In particular know:

(a) The definition of $E[Y|X]$ and be able to prove that

$E[Y] = E[E[Y|X]]$ and $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$

(b) Know (and be able to prove) that the following result for random sums follows from these formulae. If $S = \sum_{j=1}^{N} X_j$ where the $X_j's$ are i.i.d. with common mean a and common variance σ^2 and N is a random variable which is independent of the $X's$, then:

$$E[S] = aE[N], \quad Var(S) = \sigma^2 E[N] + a^2 Var(N), \quad G_S(t) = G_N(G_X(t))$$

5. Branching processes (BP).

(a) Know the definition of the BP Y_n with generating r.v. X.

(b) Be able to derive formulae for $E(Y_n)$ and $Var(Y_n)$.

(c) Be able to find the p.g.f. for Y_n for small n in simple examples using $G_{n+1}(t) = G(G_n(t))$.

(d) Be able to find $\theta_n = P(Y_n = 0)$ for small n in simple examples using $\theta_{n+1} = G(\theta_n)$.