Probability Modles. Solutions to Problem Sheet 1.

1. If $G_X(t)$ is a p.g.f. then $G_X(t)$ has a power series expansion (for $0 \le t \le 1$). The coefficients of the powers of t in the expansion give the probabilities $P(X = x)$ for $x = 0, 1, ...$ so must each be in the range $[0, 1]$ and $G_X(1) = 1$ (since the probabilities sum to 1).

(a) Here $G_X(1) = \frac{13}{12}$. So $G_X(1) \neq 1$ and $G_X(t)$ cannot be a p.g.f.

(b) $G_X(t) = \frac{1}{4}t(1+t)^2 = \frac{1}{4}t + \frac{1}{2}t^2 + \frac{1}{4}t^3$. So $G_X(t)$ has a power series expansion. The coefficient of each power of t is in the range $[0, 1]$ and the coefficients sum to one ($G_X(1) = 1$). Hence it is a p.g.f and the corresponding probability mass function has $P(X = 1) = \frac{1}{4}$, $P(X = 2) = \frac{1}{2}$, $P(X = 3) = \frac{1}{4}$ and $P(X = x) = 0$ for all other non-negative integer values of X.

(c) $G_X(t) = \frac{2t}{(1+t)}$. Although $G_X(1) = 1$, $G_X(t)$ is not a p.g.f.

You can show this by noting that $P(X = 1) = G'_X(0) = 2$ which is impossible since $0 \le P(X = x) \le 1$ for all x.

You could also look at the power series expansion, which is

$$G_X(t) = 2t\,(1+t)^{-1} = 2t\left(1 - t + t^2 - ...\right) = 2t - 2t^2 + 2t^3 - ...$$

Even non-zero powers of t have negative coefficients and all probabilities must be non-negative, also odd power have coefficients all equal to 2 and probabilities must be at most 1 so $G_X(t)$ is not a p.g.f.

(d) This can easily be expanded in a power series.

$$G_X(t) = \frac{1}{2-t} = \frac{1}{2}\left(1 - \frac{t}{2}\right)^{-1} = \frac{1}{2}\sum_{r=0}^{\infty}\left(\frac{t}{2}\right)^r = \sum_{r=0}^{\infty}\frac{1}{2^{r+1}}t^r$$

The coefficients of the powers of t all lie in the range $[0, 1]$ and $G_X(1) = 1$ so that the probabilities sum to 1. This is a p.g.f. and $P(X = x)$ is the coefficient of t^x in the power series expansion of $G_X(t)$. Hence $P(X = x) = \frac{1}{2^{x+1}}$ for $x = 0, 1, 2, ...$

You may also have noticed that $G_X(t) = \frac{p}{(1-qt)}$ (where $q = 1 - p$) with $p = \frac{1}{2}$ so it is OK to say that this is the p.g.f. of a standard distribution related to the geometric distribution. We briefly mentioned this p.g.f. in lectures. It arises when X counts the number of FAILURES before the first success for a sequence of independent trials of an experiment with probability p of success at each trial. The geometric distribution arises when we counts the number of TRIALS needed for the first success to be obtained. In lectures we showed that $P(X = x) = q^x p$ for $x = 0, 1, 2, ...$

Hence for this example we need $p = \frac{1}{2}$ and then $P(X = x) = \left(\frac{1}{2}\right)^x \frac{1}{2} = \frac{1}{2^{x+1}}$ for $x = 0, 1, 2, ...$

2. (a) $G_X(t) = e^{7t-7} = e^{7(t-1)}$. This is the p.g.f. corresponding to a Poisson distribution with parameter $\lambda = 7$. Hence $X \sim Poisson(7)$.

(b) $G_X(t) = (q + pt)^n$ where $n = 2$ and $p = \frac{1}{5}$. Hence $G_X(t)$ is the p.g.f. of a binomial distribution with parameters $n = 2$ and $p = \frac{1}{5}$. Therefore $X \sim Binomial(2, \frac{1}{5})$.

(c) Since $G_X(t) = \left(\frac{1}{5}\right)^2 (4 + t)^2 = \left(\frac{4}{5} + \frac{1}{5}t\right)^2$ this is the same p.g.f. as in (b). So $X \sim Binomial(2, \frac{1}{5})$.

3. Let X be a random variable with probability generating function

$G_X(t) = \frac{t}{4-3t} = \frac{\frac{1}{4}t}{1-\frac{3}{4}t}$. This is the p.g.f. corresponding to a geometric distribution with $p = \frac{1}{4}$. Hence $X \sim Geometric(\frac{1}{4})$.

We can differentiate $G_X(t)$ to obtain $P(X = 1)$, $P(X = 2)$, $E(X)$ and $\text{Var}(X)$.

$$G_X^{'}(t) = \frac{1}{4-3t} + \frac{3t}{(4-3t)^2}$$

Hence $E[X] = G_X^{'}(1) = 1 + 3 = 4$ and $P(X = 1) = G_X^{'}(0) = \frac{1}{4}$.

$$G_X^{(2)}(t) = \frac{3}{(4-3t)^2} + \frac{3}{(4-3t)^2} + \frac{18t}{(4-3t)^3}$$

Hence $E[X(X-1)] = G_X^{(2)}(1) = 3 + 3 + 18 = 24$ and so

$$Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = 24 + 4 - 16 = 12$$

Also $P(X = 2) = \frac{1}{2} G_X^{(2)}(0) = \frac{1}{2}\left(\frac{3}{16} + \frac{3}{16}\right) = \frac{3}{16}$.

4 (a). $G_X(t) = \left(\frac{1}{2} + \frac{1}{2}t\right)^2$ and $G_Y(t) = \left(\frac{2}{3} + \frac{1}{3}t\right)$. Hence

$$G_Z(t) = G_X(t)G_Y(t) = \left(\frac{1}{2} + \frac{1}{2}t\right)^2 \left(\frac{2}{3} + \frac{1}{3}t\right) = \frac{1}{6} + \frac{5}{12}t + \frac{1}{3}t^2 + \frac{1}{12}t^3$$

Therefore $P(Z = 0) = \frac{1}{6}$, $P(Z = 1) = \frac{5}{12}$, $P(Z = 2) = \frac{1}{3}$ and $P(Z = 3) = \frac{1}{12}$. ($P(Z = z) = 0$ for all other non-negative values of Z.)

(b). Use generating functions

$$G_Z(t) = G_X(t)G_Y(t) = e^{\lambda(t-1)}e^{\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$$

Since this is the p.g.f. corresponding to a Poisson distribution with parameter $(\lambda + \mu)$, $Z \sim Poisson(\lambda + \mu)$.

$$G_W(t) = \prod_{j=1}^{n} G_{X_j}(t) = \prod_{j=1}^{n} e^{\lambda(t-1)} = e^{n\lambda(t-1)}$$

Hence $W \sim Poisson(n\lambda)$.

(c). We differentiate $G_X(t) = \frac{(t+2t^2)}{3(2-t)}$ to obtain $E(X)$ and $E[X(X-1)]$.

$$G'_X(t) = \frac{(1+4t)}{3(2-t)} + \frac{(t+2t^2)}{3(2-t)^2}$$

Therefore $E[X] = G'_X(1) = \frac{5}{3} + 1 = \frac{8}{3}$

$$G_X^{(2)}(t) = \frac{4}{3(2-t)} + \frac{(1+4t)}{3(2-t)^2} + \frac{(1+4t)}{3(2-t)^2} + \frac{2(t+2t^2)}{3(2-t)^3}$$

Therefore $E[X(X-1)] = G_X^{(2)}(1) = \frac{4}{3} + \frac{5}{3} + \frac{5}{3} + 2 = \frac{20}{3}$ and hence $Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = \frac{20}{3} + \frac{8}{3} - \frac{64}{9} = \frac{20}{9}$

If we factor $G_X(t)$ into the product of two probability generating functions $G_X(t) = G_Y(t)G_Z(t)$, then $G_Y(t) = \frac{1}{3}(1+2t) = \left(\frac{1}{3} + \frac{2}{3}t\right)$ and $G_Z(t) = \frac{\frac{1}{2}t}{1-\frac{1}{2}t}$. Hence $Y \sim Bernoulli\left(\frac{2}{3}\right)$ and $Z \sim Geometric\left(\frac{1}{2}\right)$. (Note that you can interchange the roles of Y and Z.)

Then $E[X] = E[Y] + E[Z] = \frac{2}{3} + 2 = \frac{8}{3}$ and $Var(X) = Var(Y) + Var(Z) = \frac{2}{9} + 2 = \frac{20}{9}$.