Probability Models. Solutions to Problem Sheet 9.

1. $f_{X,Y}(x,y) = 1$ for $0 < x < 1$ and $0 < y < 1$. Let $Z = X+Y$ and $U = X$. Then $X = U$ and $Y = Z - U$. Hence

$$f_{U,Z}(u,z) = 1 \times \left\| \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right\| = 1$$

The ranges are $0 < u < 1$ and $0 < z - u < 1$. Since $U \equiv X$ we can write this as $f_{X,Z}(x,z) = 1$ for $0 < x < 1$ and $z - 1 < x < z$. (Suggestion: for better understanding of what follows draw a picture of the domain on the Z, X plain where these inequalities are satisfied.)

Either: Consider different ranges of Z, (i) $0 < z \leq 1$; (ii) $1 < z < 2$.

If $0 < z \leq 1$ then $0 < x < z$ and hence $f_Z(z) = \int_0^z dx = z$ and so $f_{X|Z}(x|z) = \frac{1}{z}$ for $0 < x < z$ so that $X|Z = z \sim U(0,z)$ for $0 < z < 1$.

If $1 < z < 2$ then $z-1 < x < 1$ and hence $f_Z(z) = \int_{z-1}^1 dx = 2 - z$ and so $f_{X|Z}(x|z) = \frac{1}{2-z}$ for $z - 1 < x < 1$ so that $X|Z = z \sim U(z-1, 1)$ for $1 < z < 2$.

Or: Write the ranges as $\max(0, z-1) < x < \min(1, z)$ and $0 < z < 2$.

Then $f_Z(z) = \min(1,z) - \max(0, z-1)$ for $0 < z < 2$ which can be written as $f_Z(z) = z$ for $0 < z \leq 1$ and $f_Z(z) = 2 - z$ for $1 < z < 2$.

Hence $f_{X|Z}(x|z) = \frac{1}{\min(1,z)-\max(0,z-1)}$ for $\max(0, z-1) < x < \min(1,z)$ so that $X|Z = z \sim U(\max(0, z-1), \min(1,z))$. This may also be written as: when $0 < z \leq 1$ then $f_{X|Z}(x|z) = \frac{1}{z}$ for $0 < x < z$ (so $X|Z = z \sim U(0,z)$) and when $1 < z < 2$ then $f_{X|Z}(x|z) = \frac{1}{2-z}$ for $z - 1 < x < 1$ (so $X|Z = z \sim U(z-1, 1)$).

2. (a) $f_U(u) = \theta e^{-\theta(u-\alpha)}$ for $\alpha < u < \infty$. Hence $V = U - \alpha$ has inverse $U = V + \alpha$ and $f_V(v) = \theta e^{-\theta v} \times |1| = \theta e^{-\theta v}$ for $v > 0$ i.e. $V \sim Exp(\theta)$. Therefore $E[U] = E[V + \alpha] = E[V] + \alpha = \frac{1}{\theta} + \alpha$ and $Var(U) = Var(V + \alpha) = Var(V) = \frac{1}{\theta^2}$. (Only statement of result needed)

(b) (i) $f_{X,Y}(x,y) = 2\theta^2 e^{-\theta(x+y)}$ for $0 < x < y < \infty$. Hence

$$f_X(x) = \int_x^\infty 2\theta^2 e^{-\theta(x+y)} dy = \left[-2\theta e^{-\theta(x+y)}\right]_{y=x}^{y=\infty} = 2\theta e^{-2\theta x}$$

for $0 < x < \infty$. Therefore $X \sim Exp(2\theta)$ and so $E[X] = \frac{1}{2\theta}$ and $Var(X) = \frac{1}{4\theta^2}$.

(ii) Then for $x > 0$, $f_{Y|X}(y|x) = \begin{cases} \frac{2\theta^2 e^{-\theta(x+y)}}{2\theta e^{-2\theta x}} = \theta e^{-\theta(y-x)} & \text{if } x < y < \infty \\ 0 & \text{otherwise .} \end{cases}$

(iii) This is the same form of p.d.f. as in part (a) with $\alpha = x$. Hence $E[Y|X] = \frac{1}{\theta} + X$ and $Var(Y|X) = \frac{1}{\theta^2}$. Therefore

$$E[Y] = E[E[Y|X]] = E\left[\frac{1}{\theta} + X\right] = \frac{1}{\theta} + \frac{1}{2\theta} = \frac{3}{2\theta}$$

$$\begin{aligned} Var(Y) &= E[Var(Y|X)] + Var(E[Y|X]) = E\left[\frac{1}{\theta^2}\right] + Var\left(\frac{1}{\theta} + X\right) \\ &= \frac{1}{\theta^2} + Var(X) = \frac{1}{\theta^2} + \frac{1}{4\theta^2} = \frac{5}{4\theta^2} \end{aligned}$$

(iv)

$$\begin{aligned} E[XY] &= E[XE[Y|X]] = E\left[X\left(\frac{1}{\theta} + X\right)\right] = \frac{1}{\theta}E[X] + (Var(X) + (E[X])^2) \\ &= \frac{1}{2\theta^2} + \frac{1}{4\theta^2} + \left(\frac{1}{2\theta}\right)^2 = \frac{1}{\theta^2} \end{aligned}$$

Hence $Cov(X,Y) = \frac{1}{\theta^2} - \left(\frac{1}{2\theta}\right)\left(\frac{3}{2\theta}\right) = \frac{1}{4\theta^2}$. Therefore

$$\rho(X,Y) = \frac{\frac{1}{4\theta^2}}{\sqrt{\frac{1}{4\theta^2} \times \frac{5}{4\theta^2}}} = \frac{1}{\sqrt{5}}$$

3. (a) Markov's inequality for a non-negative r.v. X with mean μ states that for any $h > 0$, $P(X \geq h) \leq \frac{\mu}{h}$. So here we simply take $h = \mu + 2\sigma$ to obtain

$$P(X \geq \mu + 2\sigma) \leq \frac{\mu}{\mu + 2\sigma}$$

So the upper bound for $P(X \geq \mu + 2\sigma)$ is $\frac{\mu}{\mu+2\sigma}$.

(b) If X has mean μ and variance σ^2 then Chebyshev's inequality states that, for any $h > 0$,

$$P(|X-\mu| \geq h) \leq \frac{\sigma^2}{h^2}$$

So we just need to take $h = 2\sigma$. Then Chebyshev's inequality states that

$$P(|X-\mu)| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4}$$

So the upper bound for $P(|X-\mu| \geq 2\sigma)$ is $\frac{1}{4}$.

If $X \sim Exp(\theta)$, then $\mu = \frac{1}{\theta}$ and $\sigma^2 = \frac{1}{\theta^2}$. Then:

(a) Markov's inequality is just $P\left(X \geq \frac{3}{\theta}\right) \leq \frac{1}{3}$. The exact probability is just

$$P\left(X \geq \frac{3}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$

(b) Chebyshev's inequality is just $P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) \leq \frac{1}{4}$ The exact probability is just

$$P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) = P\left(X \geq \frac{3}{\theta}\right) + P\left(X \leq -\frac{1}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$