Probability Models. Solutions to Problem Sheet 8.

1. (a) $f_{X,Y}(x,y) = 2$ for $0 < x < y < 1$. If $U = Y - X$ and $V = X$ then the inverses are $X = V = a(U,V)$ and $Y = U + V = b(U,V)$. The ranges are $0 < v < u + v < 1$, or, equivalently, $v > 0$, $u > 0$ and $u + v < 1$. Then, for u, v satisfying these inequalities we have

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \times |\det \begin{pmatrix} \frac{\partial a(u,v)}{\partial u} & \frac{\partial a(u,v)}{\partial v} \\ \frac{\partial b(u,v)}{\partial u} & \frac{\partial b(u,v)}{\partial v} \end{pmatrix}| = 2 \times |\det \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}| = 2.$$

The answer thus is

$$f_{U,V}(u,v) = \begin{cases} 2 & \text{if } v > 0,\ u > 0 \text{ and } u + v < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) When $U = u$ then V takes values v where $0 < v < 1 - u$. Note that $0 < u < 1$. Therefore $f_U(u) = \int_0^{1-u} 2dv = 2(1-u)$ if $0 < u < 1$. Hence the answer is

$$f_U(u) = \begin{cases} 2(1-u) & \text{if } 0 < u < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) To find the $Cov(U,V)$ we first find $E(U) = 2\int_0^1 u(1-u)du = \frac{1}{3}$. Note that $f_{U,V}(u,v)$ is a symmetric function of u, v and therefore $f_V(v) = \begin{cases} 2(1-v) & \text{if } 0 < v < 1, \\ 0 & \text{otherwise.} \end{cases}$ Hence $E(V) = 2\int_0^1 v(1-v)dv = \frac{1}{3}$. (In fact, one could simply say that due to the symmetry, $E(V) = \frac{1}{3}$; no calculation is necessary!) Next,

$$E(UV) = 2\int\int_{v>0,\, u>0,\, u+v<1} uvdudv = 2\int_0^1 \left[\int_0^{1-u} uvdv\right] du =$$

$$\int_0^1 u(1-u)^2 du = \frac{1}{2} - 2\frac{1}{3} + \frac{1}{4} = \frac{1}{12}.$$

Finally, $Cov(U,V) = E(UV) - E(U) \times E(V) = -\frac{1}{36}$.

2. The joint p.d.f. of X, Y is $f_{X,Y}(x,y) = \theta^2 e^{-\theta x} e^{-\theta y}$ for $x > 0$ and $y > 0$ (and 0 otherwise). Let $U = \frac{X}{Y}$ and $V = X + Y$. Inverses are found by substituting $X = YU$ so that $Y = \frac{V}{1+U}$ and hence $X = \frac{UV}{1+U} \equiv V\left(1 - \frac{1}{1+U}\right)$. The latter form is more convenient for differentiation. Then

$$f_{U,V}(u,v) = \theta^2 e^{-\theta v} \times \left\| \begin{matrix} \frac{v}{(1+u)^2} & \frac{u}{(1+u)} \\ -\frac{v}{(1+u)^2} & \frac{1}{(1+u)} \end{matrix} \right\| = \theta^2 \frac{1}{(1+u)^2} v e^{-\theta v}$$

where the ranges are $\frac{uv}{1+u} > 0$ and $\frac{v}{1+u} > 0$, so that $u > 0$ and hence $v > 0$.

Therefore the ranges are independent and the joint p.d.f. splits into a product $g(u) \times h(v)$. Hence U and V are independent and (noticing that $V \sim Gamma(\theta, 2)$),

$$f_V(v) = \theta^2 v e^{-\theta v}$$

for $v > 0$ (and 0 otherwise). Hence, for $u > 0$,

$$f_U(u) = \frac{1}{(1+u)^2}$$

(and $f_U(u) = 0$ if $u \leq 0$.)

3. (a) $f_U(u) = \theta e^{-\theta(u-\alpha)}$ for $\alpha < u < \infty$. Hence $V = U - \alpha$ has inverse $U = V + \alpha$ and $f_V(v) = \theta e^{-\theta v} \times |1| = \theta e^{-\theta v}$ for $v > 0$ i.e. $V \sim Exp(\theta)$. Therefore $E[U] = E[V + \alpha] = E[V] + \alpha = \frac{1}{\theta} + \alpha$ and $Var(U) = Var(V + \alpha) = Var(V) = \frac{1}{\theta^2}$. (Only statement of result needed)

(b) (i) $f_{X,Y}(x, y) = 2\theta^2 e^{-\theta(x+y)}$ for $0 < x < y < \infty$. Hence

$$f_X(x) = \int_x^\infty 2\theta^2 e^{-\theta(x+y)} dy = \left[-2\theta e^{-\theta(x+y)}\right]_{y=x}^{y=\infty} = 2\theta e^{-2\theta x}$$

for $0 < x < \infty$. Therefore $X \sim Exp(2\theta)$ and so $E[X] = \frac{1}{2\theta}$ and $Var(X) = \frac{1}{4\theta^2}$.

(ii) Then for $x > 0$, $f_{Y|X}(y|x) = \begin{cases} \frac{2\theta^2 e^{-\theta(x+y)}}{2\theta e^{-2\theta x}} = \theta e^{-\theta(y-x)} & \text{if } x < y < \infty \\ 0 & \text{otherwise .} \end{cases}$