Probability II. Solutions to Problem Sheet 7.

1. X has p.d.f. $f_X(x) = 2\theta x e^{-\theta x^2}$ for $x > 0$ and $f_X(x)$ is zero elsewhere.

$Y = X^2$ has inverse $X = \sqrt{Y}$. The range for Y has end-points 0 and infinity. Hence for $0 < y < \infty$,

$$f_Y(y) = f_X(\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right| = 2\theta\sqrt{y}e^{-\theta y} \times \frac{1}{2\sqrt{y}} = \theta e^{-\theta y}$$

$f_Y(y) = 0$ elsewhere. This is just the p.d.f. of $Exp(\theta)$.

2. $X \sim Exp(\theta)$, hence $f_X(x) = \theta e^{-\theta x}$ for $x > 0$.

Now $Y = 1 - e^{-\theta X} = g(X)$, so the inverse is $X = -\frac{1}{\theta}\ln(1 - Y)$. The range of X for which the p.d.f. is positive is $0 < x < \infty$. The corresponding range for Y is just $0 < y < 1$. Hence for $0 < y < 1$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| = \theta(1 - y) \times \left| \frac{1}{\theta(1 - y)} \right| = 1$$

The p.d.f. for Y is zero elsewhere. Hence $Y \sim U(0, 1)$.

3. X and Y have joint p.d.f. $f_{X,Y}(x, y) = C$ for $0 < x < 2y < 2$ and $f_{X,Y}(x, y) = 0$ elsewhere.

The support of the joint p.d.f. lies in the region between the lines $X = 0$, $2Y = X$ and $2Y = 2$, i.e. $X = 0$, $Y = \frac{X}{2}$ and $Y = 1$. The area of the support is 1. Hence $1 = C \times 1$ so that $C = 1$.

$f_X(x)$=area above the line $X = x$ within the support of the joint p.d.f. For $0 < x < 2$, the length of the line is $\left(1 - \frac{x}{2}\right)$ and hence $f_X(x) = C\left(1 - \frac{x}{2}\right) = \left(1 - \frac{x}{2}\right)$. ($f_X(x) = 0$ elsewhere.)

$f_Y(y)$=area above the line $Y = y$ within the support of the joint p.d.f. For $0 < y < 1$, the length of this line is $2y$ so that $f_Y(y) = C \times 2y = 2y$. ($f_Y(y) = 0$ elsewhere.)

4. Random variables X and Y have joint p.d.f. $f_{X,Y}(x, y) = C(x^2 + xy)$ for $0 < x < 1$, $0 < y < 1$ and $f_{X,Y}(x, y) = 0$ elsewhere. Hence

$$f_X(x) = \int_0^1 C(x^2 + xy)dy = \left[C\left(x^2 y + \frac{1}{2}xy^2\right)\right]_{y=0}^{y=1} = C\left(x^2 + \frac{1}{2}x\right)$$

for $0 < x < 1$ and $f_X(x) = 0$ elsewhere. Similarly

$$f_Y(y) = \int_0^1 C(x^2 + xy)dx = \left[C\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\right]_{x=0}^{x=1} = C\left(\frac{1}{3} + \frac{1}{2}y\right)$$

for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.

We find C by integrating either marginal p.d.f., e.g.

$$1 = \int_0^1 C\left(\frac{1}{3} + \frac{1}{2}y\right) dy = \left[C\left(\frac{1}{3}y + \frac{1}{4}y^2\right)\right]_{y=0}^{y=1} = \frac{7}{12}C$$

Hence $C = \frac{12}{7}$.

5. X and Y have joint p.d.f. $f_{X,Y}(x,y) = Ce^{-(x+y)}$ for $0 < x < y < \infty$ and $f_{X,Y}(x,y) = 0$ elsewhere. Thus

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_x^{\infty} Ce^{-x}e^{-y}dy = Ce^{-x}\left[-e^{-y}\right]_{y=x}^{y=\infty} = Ce^{-2x}$$

for $0 < x < \infty$, and $f_X(x) = 0$ elsewhere. Similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^{y} Ce^{-x}e^{-y}dx = Ce^{-y}\left[-e^{-x}\right]_{x=0}^{x=y} = Ce^{-y}\left(1 - e^{-y}\right)$$

for $0 < y < \infty$ and $f_Y(y) = 0$ elsewhere.

We find C by integrating either p.d.f. Clearly it is easiest to integrate $f_X(x)$.

$$1 = \int_0^{\infty} Ce^{-2x}dx = C\left[-\frac{1}{2}e^{-2x}\right]_{x=0}^{x=\infty} = C\frac{1}{2}$$

Hence $C = 2$.