Probability Models. Solutions to Problem Sheet 6.

1. (a)

$$\begin{aligned} M_X(t) &= \int_\alpha^\infty e^{tx} e^{-(x-\alpha)} dx \\ &= \int_\alpha^\infty e^{\alpha} e^{-(1-t)x} dx \\ &= \left[-\frac{e^\alpha}{(1-t)} e^{-(1-t)x} \right]_{x=\alpha}^{x=\infty} \\ &= \frac{e^\alpha}{(1-t)} e^{-(1-t)\alpha} = \frac{e^{\alpha t}}{(1-t)} \end{aligned}$$

(b) $M'_X(t) = \frac{dM_X(t)}{dt} = \alpha e^{\alpha t}(1-t)^{-1} + e^{\alpha t}(1-t)^{-2}$. Therefore $E[X] = M'_X(0) = \alpha + 1$.

$M''_X(t) = \alpha^2 e^{\alpha t}(1-t)^{-1} + 2\alpha e^{\alpha t}(1-t)^{-2} + 2e^{\alpha t}(1-t)^{-3}$. Therefore $E[X^2] = M''_X(0) = \alpha^2 + 2\alpha + 2$ and so $Var(X) = \alpha^2 + 2\alpha + 2 - (\alpha+1)^2 = 1$.

(c) Let $Y = (X - \alpha)$. Therefore

$$M_Y(t) = E\left[e^{(X-\alpha)t}\right] = e^{-\alpha t} E[e^{tX}] = e^{-\alpha t} M_X(t)$$

Hence $M_Y(t) = e^{-\alpha t} \frac{e^{\alpha t}}{(1-t)} = \frac{1}{1-t}$. This is the m.g.f. of an exponential random variable with parameter $\theta = 1$. Therefore, by the uniqueness of the m.g.f., $Y \sim Exp(1)$.

2. $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

$$M_Y(t) = E\left[e^{\frac{(X-\mu)t}{\sigma}}\right] = E\left[e^{\frac{-\mu t}{\sigma}} e^{\frac{t}{\sigma}X}\right] = e^{\frac{-\mu t}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$$

Therefore

$$M_Y(t) = e^{\frac{-\mu t}{\sigma}} e^{\mu\frac{t}{\sigma} + \frac{1}{2}\sigma^2\left(\frac{t}{\sigma}\right)^2} = e^{\frac{1}{2}t^2}$$

This is the m.g.f. of a normal random variable with mean zero and variance one. Therefore, by the uniqueness of the m.g.f., $Y \sim N(0,1)$.

$$M_Y(t) = e^{\frac{t^2}{2}} = \sum_{r=0}^{\infty} \frac{1}{r!}\left(\frac{t^2}{2}\right)^r = \sum_{r=0}^{\infty} \frac{1}{r!2^r} t^{2r}$$

Since all the powers are even, $E[Y^{2r+1}] = 0$ for all $r = 0, 1, ...$ Also $E[Y^{2r}]$ is the coefficient of $\frac{t^{2r}}{(2r)!}$ so that $E[Y^{2r}] = \frac{(2r)!}{r!2^r}$ for $r = 1, 2, ...$

3. (a)

$$\begin{aligned}
M_X(t) &= \int_0^{\infty} e^{tx}\frac{\theta}{2}e^{-\theta x}dx + \int_{-\infty}^{0} e^{tx}\frac{\theta}{2}e^{\theta x}dx \\
&= \int_0^{\infty} \frac{\theta}{2}e^{-(\theta-t)x}dx + \int_{-\infty}^{0} \frac{\theta}{2}e^{(\theta+t)x}dx \\
&= \left[\frac{-\theta}{2(\theta-t)}e^{-(\theta-t)x}\right]_{x=0}^{x=\infty} + \left[\frac{\theta}{2(\theta+t)}e^{(\theta+t)x}\right]_{x=-\infty}^{x=0} \\
&= \frac{\theta}{2(\theta-t)} + \frac{\theta}{2(\theta+t)} = \frac{\theta^2}{\theta^2-t^2} = \left(1-\frac{t^2}{\theta^2}\right)^{-1}
\end{aligned}$$

(b) Either: expand $M_X(t)$ in a power series, so that $M_X(t) = \sum_{r=0}^{\infty}\left(\frac{t^2}{\theta^2}\right)^r = 1+\frac{t^2}{\theta^2}+\frac{t^4}{\theta^4}+.....$ Then $E[X]$ and $E[X^2]$ are the coefficients of t and of $\frac{t^2}{2!}$, so that $E[X] = 0$ and $E[X^2] = \frac{2}{\theta^2}$ and hence $Var(X) = \frac{2}{\theta^2}$.

Or: differentiate the m.g.f. So $M_X'(t) = \frac{2t}{\theta^2}\left(1-\frac{t^2}{\theta^2}\right)^{-2}$ so that $E[X] = M_X'(0) = 0$. Also $M_X''(t) = \frac{2}{\theta^2}\left(1-\frac{t^2}{\theta^2}\right)^{-2} + 2\left(\frac{2t}{\theta^2}\right)^2\left(1-\frac{t^2}{\theta^2}\right)^{-3}$ so that $E[X^2] = M_X''(0) = \frac{2}{\theta^2}$ and hence $Var(X) = \frac{2}{\theta^2}$.

(c)

$$\begin{aligned}
M_Y(t) &= E[e^{t|X|}] = \int_0^{\infty} e^{tx}\frac{\theta}{2}e^{-\theta x}dx + \int_{-\infty}^{0} e^{-tx}\frac{\theta}{2}e^{\theta x}dx \\
&= \int_0^{\infty} \frac{\theta}{2}e^{-(\theta-t)x}dx + \int_{-\infty}^{0} \frac{\theta}{2}e^{(\theta-t)x}dx \\
&= \left[\frac{-\theta}{2(\theta-t)}e^{-(\theta-t)x}\right]_{x=0}^{x=\infty} + \left[\frac{\theta}{2(\theta-t)}e^{(\theta-t)x}\right]_{x=-\infty}^{x=0} \\
&= \frac{\theta}{2(\theta-t)} + \frac{\theta}{2(\theta-t)} = \frac{\theta}{\theta-t} = \left(1-\frac{t}{\theta}\right)^{-1}
\end{aligned}$$

This is the m.g.f. for $Exp(\theta)$. Hence, by the uniqueness of the m.g.f., $Y \sim Exp(\theta)$.

4. In the following integral take $u = x^\alpha$ and $\frac{dv}{dx} = e^{-x}$ and integrate by parts.

$$\begin{aligned}
\Gamma(\alpha+1) &= \int_0^\infty x^\alpha e^{-x} dx \\
&= \left[x^\alpha(-e^{-x})\right]_{x=0}^{x=\infty} - \int_0^\infty \alpha x^{\alpha-1}(-e^{-x})dx \\
&= 0 + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \alpha\Gamma(\alpha)
\end{aligned}$$

5. (a) $X \sim Exp(\theta)$, hence $f_X(x) = \theta e^{-\theta x}$ for $x > 0$.

Now $Y = 1 - e^{-\theta X} = g(X)$, so the inverse is $X = -\frac{1}{\theta} ln(1-Y)$. The range of X for which the p.d.f. is positive is $0 < x < \infty$. The corresponding range for Y is just $0 < y < 1$. Hence for $0 < y < 1$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right| = \theta(1-y) \times \left| \frac{1}{\theta(1-y)} \right| = 1$$

The p.d.f. for Y is zero elsewhere. Hence $Y \sim U(0,1)$.

(b) $X \sim N(0,1)$. $Y = |X|$, so that Y takes values on $[0,\infty)$. Therefore $F_Y(y) = 0$ for $y \leq 0$.

For $y > 0$, the event $Y \leq y$ is just the event $|X| \leq y$, i.e. $-y \leq X \leq y$.

Hence, for $y > 0$, $F_Y(y) = P(-y < X < y) = F_X(y) - F_X(-y)$.

The p.d.f. will be zero for $y < 0$. For $y > 0$ we can differentiate the c.d.f. above to obtain

$$f_Y(y) = f_X(y) - (-f_X(-y)) = f_X(y) + f_X(-y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} + \frac{e^{-y^2/2}}{\sqrt{2\pi}} = \left(\sqrt{\frac{2}{\pi}}\right) e^{-y^2/2}$$