Probability Models. Solution to Problem Sheet 5.

1. The distribution of $X|N = n$ is $Binomial(n, 1/2)$.

$P(X = x) = P(X = x|N = 0)P(N = 0) + P(X = x|N = 2)P(N = 2)$. Note that $P(X = 0|N = 0) = 1$ and $P(X = x|N = 0) = 0$ for $x = 1, 2$. So

$$\begin{aligned} P(X = 0) &= 1 \times P(N = 0) + \frac{1}{4}P(N = 2) = \frac{1}{4} + \frac{3}{16} = \frac{7}{16} \\ P(X = 1) &= 0 \times P(N = 0) + \frac{1}{2}P(N = 2) = \frac{3}{8} \\ P(X = 2) &= 0 \times P(N = 0) + \frac{1}{4}P(N = 2) = \frac{3}{16} \end{aligned}$$

Then $E[X] = 0 \times \frac{7}{16} + 1 \times \frac{3}{8} + 2 \times \frac{3}{16} = \frac{3}{4}$ and $Var(X) = E[X^2] - (E[X])^2 = \frac{9}{8} - \frac{9}{16} = \frac{9}{16}$.

Results in lectures give $E[Y_n]$ and $Var(Y_n)$ in terms of $\mu = E[X]$ and $\sigma^2 = Var(X)$. Note that $\mu \neq 1$.

Hence $E[Y_n] = \mu^n = \left(\frac{3}{4}\right)^n$ and

$$Var(Y_n) = \frac{\sigma^2 \mu^{(n-1)}(1 - \mu^n)}{(1 - \mu)} = \frac{9}{4}\left(\frac{3}{4}\right)^{n-1}\left(1 - \left(\frac{3}{4}\right)^n\right)$$

The number of female descendants in generation n from 100 females is just $W = \sum_{j=1}^{100} W_j$, where the W_j are independent each with the same distribution as Y_n. Hence $E[W] = 100E[Y_n] = 100 \times \left(\frac{3}{4}\right)^n$ and

$$Var(W) = 100 \times Var(Y_n) = 225 \times \left(\frac{3}{4}\right)^{n-1}\left(1 - \left(\frac{3}{4}\right)^n\right)$$

2. Since $P(X = 0) = \frac{1}{3}$ and $P(X = 2) = \frac{2}{3}$, the p.g.f. is $G_X(t) = \frac{1}{3} + \frac{2}{3}t^2$. As there is one individual in generation 0, $G_{Y_1}(t) = G_X(t)$.

(a) Since $G_{Y_2}(t) = G_{Y_1}(G_X(t)) = G_X(G_X(t))$,

$$G_{Y_2}(t) = G_X(G_X(t)) = \frac{1}{3} + \frac{2}{3}\left(\frac{1}{3} + \frac{2}{3}t^2\right)^2 = \frac{11}{27} + \frac{8}{27}t^2 + \frac{8}{27}t^4$$

Hence $P(Y_2 = 0) = \frac{11}{27}$, $P(Y_2 = 2) = \frac{8}{27}$ and $P(Y_2 = 4) = \frac{8}{27}$.

(b) Either: From part (a), $\theta_2 = P(Y_2 = 0) = \frac{11}{27}$. The probability the population will die out by generation 3 is just

$$\theta_3 = P(Y_3 = 0) = G_X(\theta_2) = \frac{1}{3} + \frac{2}{3}\left(\frac{11}{27}\right)^2 = \frac{971}{2187}$$

Or: You could use the recurrence relation $\theta_{n+1} = G_X(\theta_n)$ with $\theta_0 = 0$. Then $\theta_1 = G_X(0) = \frac{1}{3}$, so $\theta_2 = G_X(\theta_1) = G_X\left(\frac{1}{3}\right) = \frac{11}{27}$ and hence $\theta_3 = G_X(\theta_2) = G_X\left(\frac{11}{27}\right) = \frac{971}{2187}$.

(c) The probability θ that the population will eventually die out is just the smallest positive solution to $G_X(t) = t$. Solving $G_X(t) = t$ gives $2t^2 - 3t + 1 = 0$ and hence $(2t-1)(t-1) = 0$ so the roots are 1 and $\frac{1}{2}$. Hence the probability of eventual extinction is $\theta = \frac{1}{2}$.

3. $G_X(t) = \sum_{k=0}^{\infty} pq^k t^k = \frac{p}{(1-qt)}$

The probability of eventual extinction θ is just the smallest positive root of $G_x(t) = t$. Solving $G_x(t) = t$ gives $qt^2 - t + p = 0$ i.e. $(qt - p)(t - 1) = 0$. So the roots are 1 and $\frac{p}{q}$.

Hence the probability of eventual extinction is $\theta = \frac{p}{1-p}$ if $p < \frac{1}{2}$ and the probability of eventual extinction is $\theta = 1$ if $p \geq \frac{1}{2}$.

4. The probability θ that the male line of descent of a particular male will eventually die out is the smallest positive root of $G_X(t) = t$. Here $G_X(t) = \frac{1}{4} + \frac{1}{4}t + \frac{1}{2}t^2$. Then solving $G_X(t) = t$ gives $2t^2 - 3t + 1 = 0$ i.e. $(2t-1)(t-1) = 0$, so the roots are 1 and $\frac{1}{2}$. Hence the probability of eventual extinction is $\theta = \frac{1}{2}$.

(i) When there are $K = 10$ males initially we simply need to find the probability that the male line descent of each male eventually dies out. Since they act independently, this is just the product of the probabilities of eventual extinction for each of the 10 males, so is just $\theta^{10} = \left(\frac{1}{2}\right)^{10}$.

(ii) Let A be the event that the surname Earwacker will eventually die out. Then $P(A|K = k) = \theta^k = \left(\frac{1}{2}\right)^k$. So

$$P(A) = \sum_{k=0}^{\infty} P(A|K = k)P(K = k) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k P(K = k) = E\left[\left(\frac{1}{2}\right)^K\right] = G_K\left(\frac{1}{2}\right)$$