Probability Models. Solutions to Problem Sheet 4.

1. (i) $E(S) = aE(N), \ \ Var(S) = \sigma^2 E(N) + a^2 Var(N)$

(ii) The shop receives $S = \sum_{j=1}^{N} X_j$ pounds. Since $N \sim Poisson(80)$ we have: $E(N) = 80$, $Var(N) = 80$. Also, in our case $a = 20$, $\sigma^2 = 20$. Hence

$$E(S) = 20 \times 80 = 1600, \quad Var(S) = 20 \times 80 + 20^2 \times 80 = 33600.$$

2. (i) Define $Z_j = 1$ if the j^{th} message is not detected by the spam filter and $Z_j = 0$ if it is detected. Then $Z_j \sim Bernoulli(p)$ and $X = \sum_{j=1}^{Y} Z_j$. Hence, applying the above formulae gives

$$E[X] = E(Z)E(Y) = p\mu, \quad Var(X) = Var(Z)E(Y) + p^2 Var(Y) = pq\mu + p^2\mu = p\mu.$$

(We use here that $E(Y) = \mu, \ Var(Y) = \mu$ and by definition the r.v. $Z \sim Bernoulli(p)$ has the same distribution as any of Z_j; by definition $q = 1 - p$. Remember also that $E(Z) = p, \ Var(Z) = pq$.)

(ii) If $Y = y$ then $X = \sum_{j=1}^{y} Z_j$. Hence

$$E[t^X | Y = y] = E\left[\prod_{j=1}^{y} t^{Z_j}\right] = \prod_{j=1}^{y} E[t^{Z_j}] = \left(E[t^Z]\right)^y = (pt + q)^y.$$

But then $E[t^X | Y]] = (pt + q)^Y$ and

$$G_X(t) = E[E[t^X | Y]] = E[(pt + q)^Y] = G_Y(pt + q) = e^{\mu((pt+q)-1)} = e^{p\mu(t-1)}.$$

(Note that the fact that $G_Y(s) = e^{\mu(s-1)}$ is due to Y being a Poisson r.v. and is used without proof which was given a long time ago.)

(iii) But this is the p.g.f. of a Poisson r.v. with parameter $\lambda = p\mu$. Hence by the uniqueness of the p.g.f., $X \sim Poisson(p\mu)$, or, equivalently, $P(X = k) = e^{-p\mu}\frac{(p\mu)^k}{k!}$.

(iv) In this case $G_Y(t) = \sum_{k=0}^{\infty} \bar{p}\bar{q}^k t^k = \frac{\bar{p}}{1-\bar{q}t}$, $E(Y) = \frac{\bar{q}}{\bar{p}}$, $Var(Y) = \frac{\bar{q}}{\bar{p}^2}$. Hence

$$E[X] = E(Z)E(Y) = p\frac{\bar{q}}{\bar{p}}, \quad Var(X) = Var(Z)E(Y) + p^2 Var(Y) = pq\frac{\bar{q}}{\bar{p}} + p^2\frac{\bar{q}}{\bar{p}^2}.$$

To find the p.g.f. of X we note first that $E[t^X | Y = y]$ depends only on y and the distribution of Z. The calculation is therefore exactly the same as in (ii):

$$E[t^X | Y = y] = E\left[\prod_{j=1}^{y} t^{Z_j}\right] = \prod_{j=1}^{y} E[t^{Z_j}] = \left(E[t^Z]\right)^y = (pt + q)^y.$$

Hence $E[t^X|Y]] = (pt+q)^Y$ (but the distribution of this r.v. is different from the one in (ii)!). We now have

$$G_X(t) = E[E[t^X|Y]] = E[(pt+q)^Y] = G_Y(pt+q) = \frac{\bar{p}}{1-\bar{q}(pt+q)}.$$

(iii) It is clear from the above formula that $X \sim Geometric(\frac{p}{1-\bar{q}q})$

Remark. Strictly speaking, this is one of the versions of the Geometric distributions. Compare this result with the one in Notes 1: the random variable there can be presented as $Y+1$ in which case

$$G_{Y+1}(t) = G_Y(t) \times G_1(t) = G_Y(t) \times t = \frac{\bar{p}t}{1-\bar{q}t},$$

where $G_1(t) = t$ is the p.g.f. of the random variable 1.