# Probability Models. Solutions to Problem Sheet 10.

**1.** $E[\overline{X}_n] = p$ and $Var(\overline{X}_n) = \frac{p(1-p)}{n}$.

(a) Applying Chebyshev's inequality to $\overline{X}_n$, and letting $h = 0.1p$ gives

$$P(|\overline{X}_n - p| \geq 0.1p) \leq \frac{p(1-p)/n}{(0.1p)^2} = \frac{100(1-p)}{np}$$

Hence $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ provided $\frac{100(1-p)}{np} \leq 0.05$, i.e. $n \geq \frac{100(1-p)}{0.05p} = \frac{2000(1-p)}{p}$.

(b) The Central Limit Theorem implies that if $Z = \frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{p(1-p)}}$, then for $n$ large $P(Z \leq z) \doteqdot \Phi(z)$ where $\Phi$ is the c.d.f. for the $N(0,1)$ distribution. Here

$$P(|\overline{X}_n - p| \geq 0.1p) = P\left(|Z| \geq \frac{\sqrt{n}0.1p}{\sqrt{p(1-p)}}\right) \doteqdot 2\left(1 - \Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right)\right)$$

Hence we want $\Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right) \doteqdot 0.975$, so $0.1\sqrt{\frac{np}{(1-p)}} = 1.96$ and therefore $n \approxeq \frac{(19.6)^2(1-p)}{p} = 384.16\left(\frac{1}{p} - 1\right)$.