Probability Models - Notes 9

The bivariate normal distribution.

Definition. Two r.v.'s (X,Y) have a bivariate normal distribution $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ if their joint p.d.f. is

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]} \qquad (1)$$

for all x,y. The parameters μ_1,μ_2 may be any real numbers, $\sigma_1>0$, $\sigma_2>0$, and $-1\le\rho\le 1$.

It is convenient to rewrite (1) in the form

$$f_{X,Y}(x,y)=ce^{-\frac{1}{2}Q(x,y)},\ \text{ where }\ c=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}}\quad\text{and}$$

$$Q(x,y)=(1-\rho^2)^{-1}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)+\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right] \qquad (2)$$

Statement. *The marginal distributions of $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ are normal with r.v.'s X and Y having density functions*

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}},\quad f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}.$$

Proof. The expression (2) for $Q(x,y)$ can be rearranged as follows:

$$Q(x,y)=\frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_1}{\sigma_1}-\rho\frac{y-\mu_2}{\sigma_2}\right)^2+(1-\rho^2)\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]=\frac{(x-a)^2}{(1-\rho^2)\sigma_1^2}+\frac{(y-\mu_2)^2}{\sigma_2^2}, \qquad (3)$$

where $a=a(y)=\mu_1+\rho\frac{\sigma_1}{\sigma_2}(y-\mu_2)$. Hence

$$f_Y(y)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)dx=ce^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\times\int_{-\infty}^{\infty}e^{-\frac{(x-a)^2}{2(1-\rho^2)\sigma_1^2}}dx=\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}},$$

where the last step makes use of the formula $\int_{-\infty}^{\infty}e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\sqrt{2\pi}\sigma$ with $\sigma=\sigma_1\sqrt{1-\rho^2}$. □

Exercise. Derive the formula for $f_X(x)$.

Corollaries.

1. Since $X\sim N(\mu_1,\sigma_1^2)$, $Y\sim N(\mu_2,\sigma_2^2)$, we know the meaning of four parameters involved into the definition of the normal distribution, namely

$$E(X)=\mu_1,\ \ Var(X)=\sigma_1^2,\ \ E(Y)=\mu_2,\ \ Var(X)=\sigma_2^2.$$

2. $X|(Y=y)$ **is a normal r.v.** To verify this statement we substitute the necessary ingredients into the formula defining the relevant conditional density:

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_1}e^{-\frac{(x-a(y))^2}{2\sigma_1^2(1-\rho^2)}}.$$

In other words, $X|(Y=y) \sim N(a(y),(1-\rho^2)\sigma_1^2)$. Hence:

3. $E(X|Y=y)=a(y)$ or, equivalently, $E(X|Y)=\mu_1+\rho\frac{\sigma_1}{\sigma_2}(Y-\mu_2)$. In particular, we see that $E(X|Y)$ is a linear function of Y.

4. $E(XY)=\sigma_1\sigma_2\rho+\mu_1\mu_2$.

Proof. $E(XY)=E[E(XY|Y)]=E[YE(X|Y)]=E[Y(\mu_1+\rho\frac{\sigma_1}{\sigma_2}(Y-\mu_2)]=\mu_1E(Y)+$ $\rho\frac{\sigma_1}{\sigma_2}[E(Y^2)-\mu_2E(Y)]=\mu_1\mu_2+\rho\frac{\sigma_1}{\sigma_2}[E(Y^2)-\mu_2^2]=\mu_1\mu_2+\rho\frac{\sigma_1}{\sigma_2}Var(Y)=\sigma_1\sigma_2\rho+\mu_1\mu_2$. □

5. $Cov(X,Y)=\sigma_1\sigma_2\rho$. This follows from Corollary 4 and the formula $Cov(X,Y)=E(XY)-E(X)E(X)$.

6. $\rho(X,Y)=\rho$. In words: ρ is the correlation coefficient of X, Y. This is now obvious from the definition $\rho(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)var(Y)}}$.

Exercise. Show that X and Y are independent iff $\rho=0$. (This is easily seen from the joint p.d.f. of X, Y.)