

Probability Models - Notes 8

Transformations of variables

Let X and Y be jointly continuous random variables with joint p.d.f. $f_{X,Y}(x,y)$ which has support S contained in \mathbb{R}^2 . Consider random variables $U = g(X,Y)$ and $V = h(X,Y)$ which is a one to one mapping from S to D with inverses $X = a(U,V)$ and $Y = b(U,V)$ which have continuous partial derivatives. Then

$$f_{U,V}(u,v) = f_{X,Y}(a(u,v), b(u,v)) \times \left| \det \begin{pmatrix} \frac{\partial a(u,v)}{\partial u} & \frac{\partial a(u,v)}{\partial v} \\ \frac{\partial b(u,v)}{\partial u} & \frac{\partial b(u,v)}{\partial v} \end{pmatrix} \right| \quad (1)$$

for $(u,v) \in D$. You can find D by rewriting the constraints on the ranges of x and y in terms of u and v .

Example. $X \sim \text{Exp}(\theta)$ independent of $Y \sim \text{Exp}(\theta)$. Find the joint p.d.f. for $U = X + Y$ and $V = Y$.

The inverses are $X = U - V$ and $Y = V$. Also $f_{X,Y}(x,y) = \theta^2 e^{-\theta(x+y)}$ for $x > 0$ and $y > 0$. Hence using the result above:

$$f_{U,V}(u,v) = \theta^2 e^{-\theta u} \times \left| \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \right| = \theta^2 e^{-\theta u}$$

The ranges $x > 0$ and $y > 0$ become $u - v > 0$ and $v > 0$, i.e. $0 < v < u < \infty$.

Student's t distribution Let $X \sim N(0,1)$ independent of $Y \sim \chi_n^2$. Find the joint p.d.f. for $T = \frac{X}{\sqrt{Y/n}}$ and $V = Y$ and hence find the marginal p.d.f. for T .

The inverses are $X = T \sqrt{V/n}$ and $Y = V$. Also

$$f_{X,Y}(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right) \left(\frac{y^{(n/2)-1} e^{-y/2}}{2^{(n/2)} \Gamma(n/2)} \right)$$

for all x and for $y > 0$. Hence using the result above:

$$f_{T,V}(t,v) = \frac{v^{(n/2)-1} e^{-v(1+(t^2/n))/2}}{\sqrt{\pi} 2^{(n+1)/2} \Gamma(n/2)} \times \left| \begin{vmatrix} \frac{\sqrt{v}}{\sqrt{n}} & \frac{t}{2\sqrt{nv}} \\ 0 & 1 \end{vmatrix} \right| = \frac{v^{((n+1)/2)-1} e^{-v(1+(t^2/n))/2}}{\sqrt{n} \sqrt{\pi} 2^{(n+1)/2} \Gamma(n/2)}$$

The range $y > 0$ become $v > 0$, hence $-\infty < t < \infty$ and $0 < v < \infty$.

We find the p.d.f. for T by integrating out over V and using the result that a Gamma p.d.f. integrates to one.

$$\begin{aligned}
f_T(t) &= \frac{\Gamma((n+1)/2)}{\sqrt{n}\sqrt{\pi}\Gamma(n/2)(1+(t^2/n))^{(n+1)/2}} \int_0^\infty \frac{((1+(t^2/n))/2)^{(n+1)/2} v^{((n+1)/2)-1} e^{-v(1+(t^2/n))/2}}{\Gamma((n+1)/2)} dv \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n}\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}
\end{aligned}$$

for $-\infty < t < \infty$. Note that $\Gamma(1/2) = \sqrt{\pi}$ and the Beta function $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ so the p.d.f. of T is usually written as

$$f_T(t) = \frac{1}{\sqrt{n}B\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}$$

Conditional Distributions and Expectations.

Definition If $f_{X,Y}(x,y)$ is a joint p.d.f. for random variables (X,Y) then for each x for which $f_X(x) > 0$, we define the conditional p.d.f. for $Y|X = x$ by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}. \quad (2)$$

This is easily seen to be a p.d.f. for Y for each fixed value of x . Note that $f_{Y|X}(y|x) \geq 0$ and that

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x) dy = \frac{1}{f_X(x)} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = 1$$

and so the necessary conditions for a function to be a p.d.f. are satisfied. Moreover, it can be shown that under certain natural conditions

$$P(Y \in (a,b)|X = x) = \lim_{\epsilon \rightarrow 0} P(Y \in (a,b)|X \in (x-\epsilon, x+\epsilon)) = \int_a^b f_{Y|X}(y|x) dy.$$

Definition We set

$$E[g(Y)|X = x] = \int_{-\infty}^{\infty} g(y) f_{Y|X}(y|x) dy. \quad (3)$$

Denote $H(x) = E[g(Y)|X = x]$ and define $E[g(Y)|X]$ by replacing x in $H(x)$ by X :

$$E[g(Y)|X] = H(X).$$

Thus, as in the discrete case, $E[g(Y)|X]$ is a random variable; moreover it is a function of X .

The following results analogous to the ones discussed in the discrete case hold:

1. $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx$
2. $E[g(Y)] = E[E[g(Y)|X]]$ and hence $E[Y] = E[E[Y|X]]$ and $Var(Y) = E[Var(Y|X)] + Var(E[Y|X])$.
3. $M_Y(t) = E[e^{tY}] = E[E[e^{tY}|X]]$.

4. For both the discrete and continuous case a similar result to 2. holds for the expectation of a function of both X and Y , namely $E[g(X, Y)] = E[E[g(X, Y)|X]]$.

There are existence requirements for the expectations, which we assume hold. A brief proof is given for 4. when X and Y are jointly continuous random variables.

Let A denote the set of real values x for which $f_X(x) > 0$. For $x \in A$,

$$E[g(X, Y)|X = x] = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x|y) dy = \frac{1}{f_X(x)} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy \quad (4)$$

Then $E[g(X, Y)|X]$ is the function of X which takes value $E[g(X, Y)|X = x]$ when $X = x$ for all $x \in A$. Therefore

$$\begin{aligned} E[E[g(X, Y)|X]] &= \int_{x \in A} E[g(X, Y)|X = x] f_X(x) dx \\ &= \int_{x \in A} f_X(x) \left(\frac{1}{f_X(x)} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy \right) dx \\ &= \int_{x \in A} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dy dx \\ &= E[g(X, Y)] \end{aligned}$$

Examples.

1. $f_{X,Y}(x, y) = 2$ for $x > 0$, $y > 0$ and $x + y < 1$. Then $f_X(x) = 2(1 - x)$ for $0 < x < 1$ and hence $f_{Y|X}(y|x) = \frac{1}{1-x}$ for $0 < y < 1 - x$. Hence $Y|X = x \sim U(0, 1 - x)$.

2. $Y|X = x \sim N(a + bx, \sigma^2)$ and $X \sim N(\mu, \tau^2)$. We will first find $E[Y]$ and $Var(Y)$.

$E[Y|X] = a + bX$ and $Var(Y|X) = \sigma^2$. therefore $E[Y] = E[E[Y|X]] = E[a + bX] = a + b\mu$ and

$$Var(Y) = E[Var(Y|X)] + Var(E[Y|X]) = E[\sigma^2] + Var(a + bX) = \sigma^2 + b^2\tau^2$$

We will now find the m.g.f. for Y and hence obtain the distribution of Y . Now $E[e^{tY}|X]$ is just the m.g.f. of Y over the conditional distribution of $Y|X$, so $E[e^{tY}|X] = e^{(a+bX)t + \sigma^2(t^2/2)}$. Therefore

$$\begin{aligned} M_Y(t) &= E[E[e^{tY}|X]] = E\left[e^{(a+bX)t + \sigma^2(t^2/2)}\right] = e^{at + \sigma^2(t^2/2)} M_X(bt) \\ &= e^{at + \sigma^2(t^2/2)} e^{\mu(bt) + \tau^2((bt)^2/2)} = e^{(a+b\mu)t + (\sigma^2 + b^2\tau^2)(t^2/2)} \end{aligned}$$

Hence $Y \sim N(a + b\mu, \sigma^2 + b^2\tau^2)$.