Probability Models - Notes 6

Jointly Continuous Random Variables

Definition. X and Y are jointly continuous random variables if there exists a function $f_{X,Y}(x,y)$ (the joint p.d.f.) which maps $\Re^2$ into $[0, \infty)$ so that, for any measurable set A, $P((X,Y) \in A) = \int\int_{(x,y)\in A} f_{X,Y}(x,y)dxdy$.

All sets of practical interest are measurable. The joint p.d.f. is non-negative for all entries. If we consider the 3-D plot of the joint p.d.f. as a function of the entries x, y, then $P((X,Y) \in A)$ is just the volume below the p.d.f. with base the set A. Hence for any set A with area zero the corresponding probability is zero, e.g. $P(X = x, Y = y) = 0$ and $P(X + Y = c) = 0$.

(i) Joint c.d.f. $F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)dtds$. Note that the order of integration can be reversed.

(ii) From calculus (subject to differentiability constraints on the joint c.d.f.) $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial y \partial x} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$. For dx and dy small and positive, $f_{X,Y}(x,y)dxdy \approxeq P(X \in (x-dx,x], Y \in (y-dy,y])$.

(iii) $1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s,t)dsdt$.

(iv) $F_X(x) = F_{X,Y}(x,\infty)$. Differentiating with respect to x gives the result that the marginal p.d.f.'s for X can be obtained by integrating the joint p.d.f. over the entry y for Y. i.e.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

A similar result holds for Y. Note that $f_X(x)$ is just the cross-sectional area above the line $X = x$.

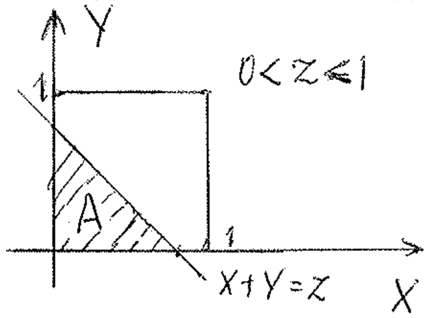
We find $P((X,Y) \in A)$ by evaluating the double integral. You have done this in Calculus 2. Remember that when you specify $f_{X,Y}(x,y)$ in the double integral it may be zero over part of the range for x and y, so make sure you put in the correct limits on the integrals corresponding to the only the values of (x,y) which are in the set A and are also in the support of the joint p.d.f.

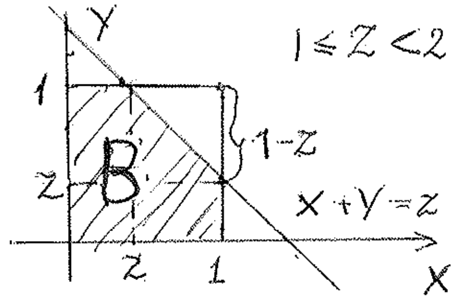
Special case with constant p.d.f. Let $f_{X,Y}(x,y) = C$ for $(x,y) \in S$ and zero elsewhere. Then from (iii), $1 = C \times (area\ of\ S)$. Hence $C = (area\ S)^{-1}$. If A is a subset of S, then $P((X,Y) \in A) = C(area\ of\ A) = \frac{area\ A}{area\ S}$.

Example. $f_{X,Y}(x,y) = C$ for $0 < x < 1$ and $0 < y < 1$. The joint p.d.f. is zero elsewhere. Then $C = 1$. The cross-sectional area above the line $X = x$ is just $C \times 1$ if $0 < x < 1$ and is zero elsewhere. Hence $f_X(x) = 1$ for $0 < x < 1$ and zero elsewhere.

Let $Z = X + Y$. We will find $P(Z \leq z) = P(X + Y \leq z)$. When $z \leq 0$, $P(Z \leq z) = 0$ and when $z \geq 2$, $P(Z \leq z) = 1$. When $0 < z \leq 1$ then $P(Z \leq z)$ corresponds to $P((X,Y) \in A)$ where A is just the interior of the triangle bounded by the X and Y axis and the line $X + Y = z$. The area of A is just $z^2/2$. Since $C = 1$, $P(Z \leq z) = z^2/2$ for $0 < z \leq 1$. Now consider $1 < z < 2$. Then $P(Z \leq z)$ corresponds to $P((X,Y) \in B)$ where B is the square forming the support of the joint p.d.f. but excluding the triangle formed by the lines $X = 1$, $Y = 1$ and $X + Y = z$. The area of this triangle is $(1-(z-1))^2/2 = (2-z)^2/2$. Hence the area of B is $(1-(2-z)^2/2)$ and so, since $C = 1$, $P(Z \leq z) = 1-(2-z)^2/2$ for $1 < z < 2$. (See the plots below.)

Therefore we have found the c.d.f. for $Z = X+Y$. Differentiating with respect to z will give the p.d.f. So $f_Z(z) = z$ for $0 < z \leq 1$, $f_Z(z) = (2-z)$ for $1 \leq z < 2$ and $f_Z(z) = 0$ elsewhere.





General case You can find the probability of an event for X and Y by integrating in either order.

Example. $f_{X,Y}(x,y) = Cxy$ for $0 < x < y < 1$ and is zero elsewhere. We will find the marginal p.d.f.'s and obtain C. Note the dependent ranges. When we find $f_X(x)$, x is held fixed and we integrate over the values of y for which the joint p.d.f. is positive, i.e. over $x < y < 1$.

$$f_X(x) = \int_x^1 Cxydy = \left[Cx\frac{y^2}{2}\right]_{y=x}^{y=1} = \frac{Cx(1-x^2)}{2}$$

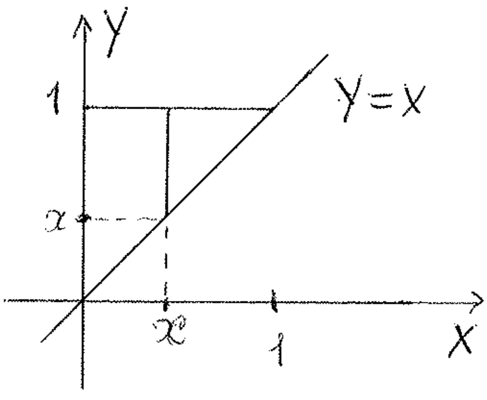
for $0 < x < 1$ and $f_X(x) = 0$ elsewhere. Integrating the p.d.f. for X identifies C.

$$1 = \int_0^1 \frac{C}{2}(x-x^3)dx = \frac{C}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = C/8$$

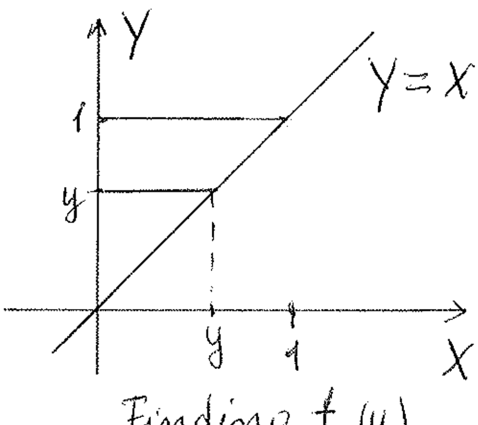
Therefore $C = 8$. When we find $f_Y(y)$, y is held fixed and we integrate over the values of x for which the joint p.d.f. is positive, i.e. over $0 < x < y$.

$$f_Y(y) = \int_0^y 8xydx = \left[8y\frac{x^2}{2}\right]_{x=0}^{x=y} = 4y^3$$

for $0 < y < 1$ and $f_Y(y) = 0$ elsewhere.



Finding $f_X(x)$



Finding $f_Y(y)$