Probability Models - Notes2

Conditional probability

Definition If A and B are events and $P(B) > 0$ then we define the conditional probability of A given B to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Definition Events $B_1, ..., B_n$ are said to partition the sample space S if $\cup_{i=1}^{n} B_i = S$ and $B_i \cap B_j = \phi$ for all $i \neq j$. (So the events are mutually exclusive and exhaustive)

The law of total probability

Let E be an event in S and let $B_1, ..., B_n$ partition S. Then

$$P(E) = \sum_{j=1}^{n} P(E|B_j)P(B_j)$$

You derived this result and looked at simple examples using this law in Probability 1

Example: A fair dice is thrown twice. Find $P(E)$ where E is the event that the product of the numbers on the die from the first and second throw is even. Let B_1 be the event that the number on the first throw is odd and B_2 be the event that the number on the first throw is even. Then B_1, B_2 form a partition of S. $P(B_1) = P(B_2) = \frac{1}{2}$. If the number on the first throw is even then the product is certain to be even. So P$(E|B_1) = \frac{1}{2}$. If the number on the first throw is odd, then the product will be even if the number on the second throw is even, so $P(E|B_2) = 1$. Hence

$$P(E) = P(E|B_1)P(B_1) + P(E|B_2)P(B_2) = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

Use of the law of total probability for sequences of independent trials or games

Consider an independent sequence of throws of a die. We throw the die until the number on the die is either a 6 or is less than or equal to 2, when we stop. Let E be the event that we stop with a throw of 6. Find $P(E)$. A useful approach is to look back to the first throw of the die. Let B_1, B_2 and B_3 correspond to the event that the first throw gives respectively 6, less than or equal to 2 and neither 6 nor less than or equal to 2. Then B_1, B_2, B_3 is a partition.

$$P(E) = P(E|B_1)P(B_1) + P(E|B_2)P(B_2) + P(E|B_3)P(B_3)$$

Now $P(B_1) = \frac{1}{6}$, $P(B_2) = \frac{1}{3}$ and $P(E_3) = \frac{1}{2}$. Also $P(E|B_1) = 1$ and $P(E|B_2) = 0$. If B_3 occurs then after the first throw we are essentially in the same situation statistically as we were at the outset. We have a sequence of independent throws and will continue until the number thrown equals 6 or is less than 2. So $P(E|B_3) = P(E)$. Hence

$$P(E) = \frac{1}{6} + P(E)\frac{1}{2}$$

Therefore solving gives $P(E) = \frac{1}{3}$.

An application of this method is the gambler's ruin problem and the random walk on a line.

The Random Walk on a Line and the Gambler's Ruin Problem.

Definition A random walk (RW) on a line is the following process. A particle moves along a line visiting only integer points. The jumps occur at integer time moments $t = 0, 1, 2, \ldots$. If at time t it is at point n then at time $t+1$ it will jump either to $n+1$ or to $n-1$. The probability of jumping from n to $n+1$ is p and the probability of jumping from n to $n-1$ is q, where $p+q = 1$. The random direction of the jump does not depend on previous jumps.

Remark In mathematical literature such RWs are often called *simple random walks*. In these notes we shall use the abbreviated term *random walk*.

Denote by X_t the coordinate of the RW at time t. The above definition means that X_t changes randomly as the time progresses according to the following rule:

$$P\{X_{t+1} = n+1 | X_t = n\} = p \quad P\{X_{t+1} = n-1 | X_t = n\} = q$$

and these probabilities do not depend on the "history" of the walk, that is to say on the values $X_0, X_1, \ldots X_{t-1}$.

Equivalently one can say that $X_{t+1} = X_t + Y_{t+1}$, where $Y_1, Y_2, \ldots$ are independent random variables each taking values 1 with probability p and -1 with probability q. Thus if X_0 is the starting position of the walk then $X_1 = X_0 + Y_1$, $X_2 = X_0 + Y_1 + Y_2$ and it is easy to see that

$$X_t = X_0 + Y_1 + Y_2 + \cdots + Y_t,$$

Exercise. Prove this relation.

We shall consider the following questions concerned with the behaviour of the RW.

1. Consider $M \le n \le N$. What is the probability that the RW starting from n would reach N before visiting M? Would reach M before N?

2. Consider $M \le n$. What is the probability that the RW starting from n would reach $+\infty$ before visiting M?

3. Suppose that $X_0 = 0$. What is the probability that the RW would ever return to 0?

4. Let $M \le n \le N$. If $X_0 = n$ denote by T_n the time at which the RW reaches either N or M. What is the expectation of T_n? (In other words what, is the mean duration of the walk?)

Before solving these problems we discuss a model which is equivalent to that of a RW.

The Gambler's Ruin Problem.

A gambler starts with a capital £k. He plays a sequence of games. At each game he has a probability p of winning and probability $q = 1 - p$ of losing the game. If he wins then he receives £1 and if he loses then he pays £1. He decides that he will stop if his capital either grows to £N or declines to £M (in the original problem $M = 0$ so he goes broke, i.e. is ruined). What is the probability that at the end of the series of games his capital would be £M? Would be £N? If he plays against a casino, what is the probability that he wins an infinite sum of money?

To understand the relation between these two models note that if X_t is the gambler's capital after t games, then $X_{t+1} = X_t + Y_{t+1}$, where Y_t is a random variable which represents the amount he "gains" in the $(t+1)$st game, which is £1 with probability p and £(-1) with probability q. Thus X_t is a random variable which behaves exactly like the coordinate of the random walk in the model considered above. In particular the probability that at the end of the series of games his capital would be £N is the same as the probability that a particle starting from k would reach N before M.

Exercise. State the analogue of questions 2 and 4 in terms of the gambler's Ruin Problem.

Solutions.

QUESTION 1. Let A_n be the event that the RW starting from n reaches N before visiting M. Set $r_n = P(A_n)$.

Theorem *The probabilities r_n, $M \le n \le M$ satisfy the following system of equations:*

$$r_n = pr_{n+1} + qr_{n-1}, \quad \text{if } M < n < N \tag{1}$$
$$r_M = 0, \quad r_N = 1 \tag{2}$$

Proof We condition on the outcome of the first jump, so B_1 and B_2 are the events 'the first jump is to the right' and 'the first jump is to the left' respectively. Then $P(B_1) = p$, $P(B_2) = q$. Next, B_1, B_2 form a partition and we therefore can use the Total Probability Law:

$$P(A_n) = P(B_1)P(A_n|B_1) + P(B_2)P(A_n|B_2)P(B_2) \tag{3}$$

If B_1 occurs then the walk jumps to $n+1$ and the process is in the same situation as initially but is now starting from $n+1$. Hence $P(A_n|B_1) = P(A_{n+1}) = r_{n+1}$. Similarly, if B_2 occurs then the walk jumps to $n-1$ and therefor $P(A_n|B_2) = P(A_{n-1}) = r_{n-1}$. Therefore equation (3) can be rewritten as

$$r_n = pr_{n+1} + qr_{n-1}$$

which proves that the main equation (1) in the statement of the Theorem holds. To prove (2) remember the walk stops immediately when it reaches M or N. Hence the probability to reach N starting from N is one: $r_N = 1$. And the probability to reach N starting from M is zero: $r_M = 0$. The Theorem is proved. □

SOLVING EQUATIONS (1), (2). Equation (1) is just a simple second order difference equation. Its partial solutions can be found in the form $r_n = \theta^n$, where θ satisfies the associated quadratic equation $p\theta^2 - \theta + q = 0$.

If the latter equation has two distinct roots $\theta_1 \neq \theta_2$ then the general solution to (1) is given by $r_n = a\theta_1^n + b\theta_2^n$.

If $\theta_1 = \theta_2$ then the general solution to (1) is given by $r_n = (a+bn)\,\theta_1^n$.

We note that in our case the roots are $\theta_1 = 1$ and $\theta_2 = \frac{q}{p}$. The roots will be equal if $p = q = \frac{1}{2}$. We thus have to consider two cases.

Case when $p \neq \frac{1}{2}$. The solution to the difference equation is

$$r_n = a(1)^n + b\left(\frac{q}{p}\right)^n = a + b\left(\frac{q}{p}\right)^n$$

Since $0 = r_M = a + b\left(\frac{q}{p}\right)^M$ and $1 = r_N = a + b\left(\frac{q}{p}\right)^N$, we obtain the solution

$$r_n = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} \tag{4}$$

Case $p = \frac{1}{2}$. The solution to the difference equation is $r_n = (a+bn)(1)^n = a+bn$. Since $0 = r_M = a + bM$ and $1 = r_N = a + bN$, we obtain the solution

$$r_n = \frac{n-M}{N-M} \tag{5}$$

Similarly, let F_n be the event that the walk reaches M before N and set $l_n = P(F_n)$. Then l_n satisfies the same difference equation as r_n, namely $l_n = pl_{n+1} + ql_{n-1}$ if $M < n < N$. However, the boundary conditions are different: $l_M = 1$ and $l_N = 0$.

Case when $p \neq \frac{1}{2}$. The solution is

$$l_n = \frac{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^n}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} \tag{6}$$

Remark. Note that $r_n + l_n = 1$ so the series of games are certain to finish.

Case $p = \frac{1}{2}$. The solution is

$$l_n = \frac{N-n}{N-M} \tag{7}$$

Remark. Again $r_n + l_n = 1$.

If we indicate in the notation the boundaries M and N then we replace r_n by $r_n(M,N)$ and l_n by $l_n(M,N)$ in the results above.

QUESTION 2. To answer this question we observe first that

$$r_n(M,\infty) \stackrel{\text{def}}{=} P\{\text{reach } +\infty \text{ before } M\} = \lim_{N\to+\infty} r_n(M,N).$$

We write $r_n(M,N)$ to emphasize the dependence in (4) on n, M, and N. However, n, M will be fixed while $N \to +\infty$. The answer depends on the relation between p and q.

Case when $p > q$. Then $\frac{q}{p} < 1$ and

$$\lim_{N\to+\infty}\left(\frac{q}{p}\right)^N = 0.$$

Therefore

$$\lim_{N\to+\infty} r_n(M,N) = \lim_{N\to+\infty} \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{0 - \left(\frac{q}{p}\right)^M} = 1 - \left(\frac{q}{p}\right)^{n-M}.$$

Case when $p<q$. Then $\frac{q}{p} > 1$ and

$$\lim_{N\to+\infty} \left(\frac{q}{p}\right)^N = \infty.$$

Therefore

$$\lim_{N\to+\infty} r_n(M,N) = \lim_{N\to+\infty} \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M} = \frac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\infty} = 0.$$

Case when $p=q$. Then one obtains from (5) that

$$\lim_{N\to+\infty} r_n(M,N) = \lim_{N\to+\infty} \frac{n-M}{N-M} = 0.$$

The three cases can be summarized as follows:

$$r_n(M,\infty) = \begin{cases} 1 - \left(\frac{q}{p}\right)^{n-M} & \text{if } q<p \\ 0 & \text{othewise .} \end{cases}$$

Exercise. Prove that

$$l_n(-\infty,N) \stackrel{\text{def}}{=} P\{\text{reach } -\infty \text{ before } N\} = \lim_{M\to-\infty} l_n(M,N) = \begin{cases} 1 - \left(\frac{p}{q}\right)^{N-n} & \text{if } p<q \\ 0 & \text{othewise} \end{cases}$$

and

$$r_n(-\infty,N) \stackrel{\text{def}}{=} P\{\text{reach } N \text{ before } -\infty\ \} = \lim_{M\to-\infty} r_n(M,N) = \begin{cases} \left(\frac{p}{q}\right)^{N-n} & \text{if } p<q \\ 1 & \text{othewise .} \end{cases}$$

QUESTION 3. Let B_1, B_2 be the events that a RW starting from 0 makes its first step to the right or left respectively. Let R denote the event that the RW returns to 0. Then by TPF

$$P(R) = P(B_1)P(R|B_1) + P(B_2)P(R|B_2) = pl_1(0,+\infty) + qr_{-1}(-\infty,0).$$

The above formulae allow us to solve the problem immediately. However, we need one more ingredient, namely $l_1(0,+\infty)$. One can either compute it directly (as above) or to use the relation $l_1(0,+\infty) = 1 - r_1(0,+\infty)$. As before, the answer will depend on the relation between p and q. E.g. suppose that $p<q$. Then $l_1(0,+\infty) = 1 - r_1(0,+\infty) = 1 - 0 = 1$. Next, $r_{-1}(-\infty,0) = \left(\frac{p}{q}\right)^{N-n} = \left(\frac{p}{q}\right)^{0-(-1)} = \frac{p}{q}$. Thus $P(R) = p\times 1 + q \times \frac{p}{q} = 2p$. Similarly, if $p>q$, then $P(R) = 2q$. Note that $P(R) = 1$ if and only if $p=q=0.5$. Summarizing, we can state that

$$P(R) = \begin{cases} 2p & \text{if } p<q, \\ 2q & \text{othewise .} \end{cases}$$