Probability Models - Notes 11

Poisson Process with intensity λ.

Informally, a random process can be defined as a family of random variables $X(t)$ depending on a real parameter t. Here we shall consider the case when t is a non-negative parameter interpreted as time. A stream of phone calls passing through a telephone exchange at a big hotel is a popular example of such a process. A Poisson Process with intensity λ is one of the simplest mathematical models describing this real life situation.

Definition A random process $X(t)$ is called a Poisson Process with intensity $\lambda > 0$ if

1. $X(0) = 0$.

2. For any $t \geq 0$ and $s \geq 0$ the random variable $X(t+s) - X(t)$ has a Poisson distribution with parameter λs:

$$P\{X(t+s) - X(t) = k\} = e^{-\lambda s}\frac{(\lambda s)^k}{k!}, \quad k = 0, 1, 2, \dots.$$

3. If $0 = t_0 < t_1 < \dots < t_n$ is a sequence of moments of time then $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$, ... , $X(t_n) - X(t_{n-1})$ are mutually independent random variables.

Exercise. Prove that $X(t)$ is a monotone function of t taking integer values.

Note that it follows from the above definition that $X(t)$ is a Poisson random variable with parameter λt. Hence $E[X(t)] = \lambda t$ and $\lambda = \frac{E[X(t)]}{t}$. We thus see that λ is the mean number of events one expects to observe during one unit of time.

We shall now discuss several properties of the Poisson Process. These properties will be presented as answers to some natural questions which can be asked about a Poisson Process.

Question 1. What is the joint distribution of the values of the Poisson process at times $t_1, t_2, \dots, t_n$, where $t_1 < t_2 < \dots < t_n$?

Solution. We need to find $P\{X(t_1) = k_1, X(t_2) = k_2, \dots, X(t_n) = k_n\}$. Note that this probability is not equal to 0 only if $k_1 \leq k_2 \leq \dots \leq k_n$ (explain this).

If $n = 1$ then $P\{X(t_1) = k_1\} = P\{X(t_1) - X(0) = k_1\} \overset{\text{by 2.}}{=} e^{-\lambda t_1}\frac{(\lambda t_1)^{k_1}}{k_1!}$.

If $n = 2$ then

$$\begin{aligned}
P\{X(t_1) = k_1, X(t_2) = k_2\} &= P\{X(t_1) = k_1, X(t_2) - X(t_1) = k_2 - k_1\} \\
&\overset{\text{by 3.}}{=} P\{X(t_1) = k_1\} \times P\{X(t_2) - X(t_1) = k_2 - k_1\} \\
&= e^{-\lambda t_1}\frac{(\lambda t_1)^{k_1}}{k_1!} \times e^{-\lambda(t_2 - t_1)}\frac{(\lambda(t_2 - t_1))^{k_2 - k_1}}{(k_2 - k_1)!} = e^{-\lambda t_2}\lambda^{k_2}\frac{t_1^{k_1}(t_2 - t_1)^{k_2 - k_1}}{k_1!(k_2 - k_1)!}.
\end{aligned}$$

Similarly, it is easy to show that

$$P\{X(t_1) = k_1, X(t_2) = k_2, \dots, X(t_n) = k_n\} = e^{-\lambda t_n}\lambda^{k_n}\frac{t_1^{k_1}(t_2 - t_1)^{k_2 - k_1}\dots(t_n - t_{n-1})^{k_n - k_{n-1}}}{k_1!(k_2 - k_1)!\dots(k_n - k_{n-1})!}.$$

$X(t)$ can be interpreted as the number of events which took place by time t (e.g. the number of calls received by a telephone exchange). It is natural to consider the random times at which these events occur: T_1, T_2, ..., $T_n, \ldots$. These times are called *the waiting times* or the *the arrival times*, sometimes *the occurrence times*. Naturally, *the inter-arrival times* are the random intervals of time which separate consecutive events: $W_1 \stackrel{\text{def}}{=} T_1$, $W_2 \stackrel{\text{def}}{=} T_2 - T_1, \ldots$, $W_n \stackrel{\text{def}}{=} T_n - T_{n-1}, \ldots$

Question 2. What is the distribution of T_n?

Solution. We need to find the p.d.f. $f_{T_n}(x)$ of the r.v. T_n. To this end we shall first find the cumulative distribution function $F_{T_n}(x) = P\{T_n \le x\} = 1 - P\{T_n > x\}$ and then use the formula $f_{T_n}(x) = F'_{T_n}(x)$. Note that $T_n \ge 0$ and therefore $F_{T_n}(x) = 0$ if $x < 0$. Thus we only need to consider $x \ge 0$.

We start with $n = 1$ (and $x \ge 0$). Then we have the following equality of events: $\{T_1 > x\} = \{X(x) = 0\}$. Hence $P\{T_1 > x\} = P\{X(x) = 0\} = e^{-\lambda x}$ (by property 2. of a Poisson Process). Hence the distribution function of T_1 is $F_{T_1}(x) = 1 - e^{-\lambda x}$ and therefore $f_{T_1}(x) = (1 - e^{-\lambda x})' = \lambda e^{-\lambda x}$. The answer thus is that T_1 is an exponential r. v. with parameter λ:

$$f_{T_1}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Next, let $n = 2$. Then $\{T_2 > x\} = \{X(x) = 0\} \cup \{X(x) = 1\}$ and $P\{T_2 > x\} = P\{X(x) = 0\} + P\{X(x) = 1\} = e^{-\lambda x} + e^{-\lambda x}\lambda x$. Hence $f_{T_2}(x) = (1 - e^{-\lambda x} - e^{-\lambda x}\lambda x)' = \lambda^2 x e^{-\lambda x}$. The answer thus is

$$f_{T_2}(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Similarly, for any $n \ge 1$ we have $P\{T_n > x\} = P\{X(x) \le n-1\}$ and hence

$$P\{T_n > x\} = P\{X(x) = 0\} + P\{X(x) = 1\} + \ldots + P\{X(x) = n-1\} = \sum_{k=0}^{n-1} e^{-\lambda x}\frac{(\lambda x)^k}{k!}.$$

Hence $f_{T_n}(x) = (1 - \sum_{k=0}^{n-1} e^{-\lambda x}\frac{(\lambda x)^k}{k!})' = \lambda^n \frac{x^{n-1}}{(n-1)!}e^{-\lambda x}$. This means that $T_n \sim Gamma(\lambda, n)$ with

$$f_{T_n}(x) = \begin{cases} \lambda^n \frac{x^{n-1}}{(n-1)!}e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

We now turn to the distribution of W_n. Since $W_1 = T_1$ we have that

$$f_{W_1}(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

which means that $W_1 \sim Exponential(\lambda)$. It turns out that the following theorem holds:

Theorem. *The inter-arrival times are mutually independent random variables each having exponential distribution with parameter λ, that is*

$$f_{W_1,W_2,\ldots,W_n}(x_1,x_2,\ldots,x_n) = \begin{cases} \lambda^n e^{-\lambda(x_1+x_2+\ldots+x_n)} & \text{if } x_1 > 0, x_2 > 0, \ldots, x_n > 0 \\ 0 & \text{otherwise .} \end{cases}$$

The proof of this theorem has not been explained in lectures and you are not supposed to know it. However, it is a very good exercise to prove this statement for $n = 2$.